AN

ELEMENTARY TREATISE

ON

PURE GEOMETRY

WITH NUMEROUS EXAMPLES

BY

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PREFACE TO THE FIRST EDITION

In this treatise, the author has attempted to bring together all the well-known theorems and examples connected with Harmonics, Anharmonics, Involution, Projection (including Homology), and Reciprocation. In order to avoid the difficulty of framing a general geometrical theory of Imaginary Points and Lines, the Principle of Continuity is appealed to. The properties of Circular Points and Circular Lines are then discussed, and applied to the theory of the Foci of Conics.

The examples at the ends of the articles are intended to be solved by the help of the article to which they are appended. Among these examples will be found many interesting theorems which were not considered important enough to be included in the text. At the end of the book there is, besides, a large number of Miscellaneous Examples. Of these, the first part is taken mainly from examination papers of the University of Oxford. Scattered throughout the book will be found examples taken from that admirable collection of problems called *Mathematical Questions and Solutions from the 'Educational Times.'* For permission to make use of these, I am indebted to the kindness of the able editor, Mr. W. J. C. Miller, B.A., Registrar of the General Medical Council.

The book has been read both in MS. and in proof by my old pupil, Mr. A. E. Jolliffe, B.A., Fellow of Corpus Christi College, and formerly Scholar of Balliol College, Oxford, whose valuable suggestions I have made free use of. To him I am also indebted for the second part of the Miscellaneous Examples. I am glad of this opportunity of acknowledging my great obligations to my former tutor, the late Professor H. J. S. Smith. My first lessons in Pure Geometry were learnt from his lectures; and many of the proofs in this book are derived from the same source.

I have assumed that the reader has passed through the

ordinary curriculum in Geometry before attempting to read the present subject; viz. Euclid, some Appendix to Euclid, and Geometrical Conics.

I have not found it convenient to keep rigidly to any single notation. But, ordinarily, points have been denoted by $A, B, C, \ldots$, lines by $a, b, c, \ldots$, and planes and conics by $\alpha, \beta, \gamma, \ldots$.

The following abbreviations have been used—

A straight line has been called a *line*, and a curved line has been called a *curve*.

The point of intersection of two lines has been called the *meet* of the lines.

The line joining two points has been called the *join* of the points.

The meet of the lines AB and CD has been denoted by $(AB; CD)$.

To avoid the frequent use of the phrase 'with respect to' or 'with regard to,' the word 'for' has been substituted.

The abbreviation 'r. h.' has sometimes been used for 'rectangular hyperbola.'

The single word 'director' has been used to include the 'director circle' of a central conic and the 'directrix' of a parabola.

The angle between the lines a and b has been denoted by $\angle ab$ and the sine of this angle by $\sin ab$.

The length of the perpendicular from the point A on the line b has been denoted by (A, b).

I have ventured to use the word 'mate' to mean 'the point (or line) corresponding.' I have avoided using the word 'conjugate' except in connexion with the theory of polars.

I shall be glad to receive, from any of my readers, corrections, or suggestions for the improvement of the book; interesting theorems and examples which are not already included will also be welcomed.

J. W. RUSSELL.

February, 1893.

PREFACE TO THE SECOND EDITION

BESIDES numerous small improvements throughout, the following are the principal changes made in this edition.

The examples in each chapter have been rearranged, the redundant ones being omitted and those of the nature of problems being placed at the end of the chapter.

Each chapter has been made independent of the following chapters; for instance in Chapter VI, proofs have been introduced that the projection of a conic is a conic, and that five points determine a conic, and in Chapter VII an elementary theory of foci has been added.

More use has been made of Projection in proofs of theorems; and correlative theorems have been proved by Reciprocation.

The method of (1, 1) correspondence has been introduced wherever possible, but only as an alternative method; as the author thinks it, although elegant and powerful, a dangerous method on account of the difficulty of seeing whether a given construction is 'rational.'

A model to illustrate figures in projection has been added at the end of the book; it is hoped that this will enable the reader to follow the theory of Projection more clearly.

The more difficult portions have been marked with a *.

Also an index has been added.

I thank Mr. G. G. Berry, of Balliol College, for kindly verifying the new matter, and numerous correspondents who have been good enough to send me corrections of the old edition and suggestions for the new. Further help in this direction will be welcomed; and the author will be glad to communicate with readers on any matter arising out of the book.

J. W. RUSSELL.

July, 1905.

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ORDINARY COURSE

Read everything except the paragraphs and chapters marked with a *.

ELEMENTARY COURSE

Omit the marked paragraphs and also :—II (4, 8) ; III (6) ; IV (6, 7) ; V (9) ; VI (6) ; VII (3, 5, 8, 9) ; VIII (13, 17, 19, 20, 21) ; IX (5, 10, 17, 18) ; X (5) ; XI (6, 9, 11, 12) ; XII (4) ; XIII (2, 3, 4, 5) ; XIV (3) ; XV (3, 5) ; XVI (5) ; XVII (3, 4, 5, 10, 13) ; XVIII (7) ; XIX (6, 7, 8) ; XX (3, 4, 6) ; XXI (4, 5, 6, 7) ; XXII (all) ; XXIII (4, 6) ; XXIV (6, 9) ; XXV (2, 3, 4, 10, 11, 12, 16) ; XXVI (2, 3) ; XXVII (all) ; XXVIII (2, 6, 9) ; XXIX (7, 9).

ABBREVIATED COURSE

Many readers will prefer to use Analytical Geometry in the following parts of the book :—II (9) ; III (7, 9) ; V (10, 14, 15, 17) ; VI (2–6, 10, 11) ; VII (2, 5, 8, 9) ; VIII (14, 16) ; XI (3, 5, 6, 7, 9, 10) ; XII (2, 4, 5, 6) ; XIII (1, 4) ; XIX (5) ; XXI (2, 3) ; XXII (all) ; XXVII (4, 5, 6, 7, 8, 9) ; XXVIII (11).

PURE GEOMETRY

CHAPTER I

FORMULAE CONNECTING SEGMENTS OF THE SAME LINE

1. ONE of the differences between Modern Geometry and the Geometry of Euclid is that a length in Modern Geometry has a sign as well as a magnitude. Lengths measured on a line in one direction are considered positive and those measured in the opposite direction are considered negative. Thus if AB, i.e. the segment extending from A to B, be considered positive, then BA, i.e. the segment extending from B to A, must be considered negative. Also AB and BA differ only in sign. Hence we obtain the first formula, viz. $AB = -BA$.

Notice that by allowing lengths to have a sign as well as a magnitude, we are enabled to utilize the formulae of Algebra in geometrical investigations. In making use of Algebra it is generally best to reduce all the segments we employ to the same origin. This is done in the following way.

$O \qquad\qquad A \qquad B$

Take any segment AB on a line and also any origin O. Then $AB = OB - OA$. This is obviously true in the above figure, and it is true for any figure. For

$$OB - OA = OB + AO = AO + OB = AB;$$

for $AO + OB$ means that the point travels from A to O and then from O to B, and thus the point has gone from A to B.

The fundamental formulae then are

(1) $AB = -BA$; (2) $AB = OB - OA$.

In the above discussion the lengths have been taken on a line. But this is not necessary; the lengths might have been taken on any curve.

It is generally convenient to use an abridged form of the formula $AB = OB - OA$, viz. $AB = b - a$, where $a = OA$ and $b = OB$.

2. *A, B, C, D are any four collinear points; show that*
$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

Take A as origin, then $CD = AD - AC = d - c$, and so on. Hence

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = b(d-c) + c(b-d) + d(c-b)$$
$$= bd - bc + cb - cd + dc - db = 0.$$

Ex. 1. *A, B, C, D, O are any five points in a plane; show that*
$$\Delta AOB \,.\, \Delta COD + \Delta AOC \,.\, \Delta DOB + \Delta AOD \,.\, \Delta BOC = 0,$$
where ΔAOB denotes the area of the triangle AOB.

Let a line meet OA, OB, OC, OD in A', B', C', D'. Then
$$\Delta AOB = \tfrac{1}{2} \,.\, OA \,.\, OB \sin AOB.$$
Hence the given relation is true if
$$\Sigma \{\sin AOB \,.\, \sin COD\} = 0,$$
i.e. if
$$\Sigma \{\sin A'OB' \,.\, \sin C'OD'\} = 0.$$
But $p \,.\, A'B' = OA' \,.\, OB' \sin A'OB'$, where p is the perpendicular from O on $A'B'C'D'$. Hence the given relation is true if $A'B' \,.\, C'D' + A'C' \,.\, D'B' + A'D' \,.\, B'C' = 0$.

Ex. 2. *If OA, OB, OC, OD be any four lines meeting in a point, show that*
$$\sin AOB \,.\, \sin COD + \sin AOC \,.\, \sin DOB + \sin AOD \,.\, \sin BOC = 0.$$

Ex. 3. *From Ex. 2 deduce Ptolemy's Theorem connecting four points on a circle.*

Take O also on the circle. Then $AB = 2 \,.\, R \,.\, \sin AOB$.

Ex. 4. *If O, A, B are any collinear points, then*
$$OA^2 + OB^2 = AB^2 + 2OA \,.\, OB.$$

Ex. 5. *If $A, B, C, \ldots X, Y$ are any number of collinear points, show that* $AB + BC + \ldots + XY + YA = 0$.

Ex. 6. *If λ denotes the ratio $OA : OB$ and λ' the ratio $OA' : OB'$ (O, A, B, A', B' being collinear points), show that*
$$BB' \,.\, \lambda \,.\, \lambda' + A'B \,.\, \lambda + B'A \,.\, \lambda' + AA' = 0.$$

Ex. 7. *If D, E, F are any three points on the sides BC, CA, AB of a triangle, show that*
$$\frac{DB \,.\, EC \,.\, FA}{DC \,.\, EA \,.\, FB} = \frac{\sin DAB \,.\, \sin EBC \,.\, \sin FCA}{\sin DAC \,.\, \sin EBA \,.\, \sin FCB}.$$

Stewart's Theorem.

3. *A, B, C are any three collinear points, and P is any other point; show that*

$$PA^2 . BC + PB^2 . CA + PC^2 . AB + BC . CA . AB = 0.$$

Drop the perpendicular PO from P on ABC.

Take A as origin (so that the factors CA and AB may be obvious); and let $AO = x$ and $PO = p$.

Then $PA^2 . BC + PB^2 . CA + PC^2 . AB$

$$= (OA^2 + OP^2) BC + (OB^2 + OP^2) CA + (OC^2 + OP^2) AB$$
$$= (x^2 + p^2)(c - b) + [(b-x)^2 + p^2](-c) + [(c-x)^2 + p^2] b$$
$$= x^2c - x^2b + (b^2 - 2bx + x^2)(-c) + (c^2 - 2cx + x^2) b$$
$$+ p^2 (c - b - c + b)$$
$$= -b^2c + c^2b = bc(c-b) = AB . AC . BC$$
$$= -BC . CA . AB.$$

Ex. 1. *If A, B, C be three collinear points and a, b, c be the tangents from A, B, C to a given circle, then*

$$a^2 . BC + b^2 . CA + c^2 . AB + BC . CA . AB = 0.$$

Ex. 2. *If P be any point on the base AB of the triangle ABC, then $AP . CB^2 - BP . CA^2 = AB . (CP^2 - AP . BP)$.*

Ex. 3. *If A, B, C, D be four points on a circle and P any point, show that*

$$\Delta BCD . AP^2 - \Delta CDA . BP^2$$
$$+ \Delta DAB . CP^2 - \Delta ABC . DP^2 = 0,$$

disregarding signs.

Let AC, BD meet in O inside the circle.

Then $\Delta BCD \propto BD . CO$ and $BO . OD = CO . OA$.

Ex. 4. *If VA, VB, VC, VD be any four lines through V, then*

$$\frac{\sin BVD . \sin CVD}{\sin BVA . \sin CVA} + \frac{\sin CVD . \sin AVD}{\sin CVB . \sin AVB}$$
$$+ \frac{\sin AVD . \sin BVD}{\sin AVC . \sin BVC} = 1.$$

Draw a parallel to VD.

Ex. 5. *If from any point P there be drawn the perpendicular PQ on the line AB, then*

$$PA^2 - PB^2 = AB^2 + 2 . AB . BQ.$$

4. *If C be the middle point of AB, then whatever origin O be chosen on the line AB, we have $OC = \frac{1}{2}(OA + OB)$.*

For $OC = OA + AC = OA + \frac{1}{2}AB = OA + \frac{1}{2}(OB - OA)$

$$= \tfrac{1}{2}(OA + OB).$$

As we have used general formulae throughout this proof, the formula holds for every relative position of the points O, A, and B.

Ex. 1. *If AA', BB', CC' be collinear segments whose middle points are α, β, γ, and if P be a variable point on the line, show that*

$PA \,.\, PA' \,.\, \beta\gamma + PB \,.\, PB' \,.\, \gamma\alpha + PC \,.\, PC' \,.\, \alpha\beta$ *is constant.*

Take any origin O. Then

$$2 \,.\, \beta\gamma = 2 \,.\, O\gamma - 2 \,.\, O\beta = c + c' - b - b'.$$

Twice the given expression is

$$(a - p)(a' - p)(c + c' - b - b') + \ldots + \ldots,$$

which is equal to $aa'(c + c' - b - b') + \ldots + \ldots$.

Ex. 2. *If C be the middle point of AB, and O be any point on the line ACB, show that*

$$OA^2 + OB^2 = CA^2 + CB^2 + 2 \,.\, OC^2.$$

Ex. 3. *If P be the middle point of the segment AA' and Q be the middle point of the segment BB' (on the same line as AA'), show that*

$$2 \,.\, PQ \,.\, AA' = AB \,.\, AB' - A'B \,.\, A'B'.$$

Ex. 4. *If on the line AB the point G be taken such that $a \,.\, GA + b \,.\, GB = 0$, a and b being any numbers, positive or negative, then, O being also on AB,*

$$a \,.\, OA + b \,.\, OB = (a + b) \,.\, OG.$$

Ex. 5. *If on the line $ABCD \ldots$ a point G be taken such that*

$$GA + GB + GC + \ldots = 0,$$

and O be any other point on the line, then

$$OA^2 + OB^2 + OC^2 + \ldots = GA^2 + GB^2 + GC^2 + \ldots + n \,.\, GO^2,$$

n being the number of the points $ABCD\ldots$.

***5.** The following is an interesting application of Algebra to Geometry:

If A, B, C, D, P, Q be any six collinear points, then

$$\frac{AP.AQ}{AB.AC.AD} + \frac{BP.BQ}{BC.BD.BA} + \frac{CP.CQ}{CD.CA.CB} + \frac{DP.DQ}{DA.DB.DC} = 0.$$

Put X for A, and reduce the resulting equation to any origin, after getting rid of the denominators. We shall have an equation of the second order in x to determine X.

Put $x = b$, i.e. $X = B$, and we get an identity.

Hence $x = b$ is one solution of this equation.

Similarly $x = c$, and $x = d$ are solutions.

Hence the equation of the second order has three solutions; and is therefore an identity.

If A, B, C, P, Q be any five collinear points, then

$$\frac{AP.AQ}{AB.AC} + \frac{BP.BQ}{BC.BA} + \frac{CP.CQ}{CA.CB} = 1.$$

Multiply the identity just proved by AD throughout and let D be at infinity.

Then $\quad AD = AB + BD, \quad \therefore \quad AD/BD = AB/BD + 1.$

But when D is at infinity $\quad AB/BD = 0.$

Hence $\quad AD/BD = 1.\quad$ Similarly $\quad AD/CD = 1.$

So $\quad DP/DB = 1 \quad$ and $\quad DQ/DC = 1.$

Hence we obtain the result enunciated.

If A, B, C, D, P be any five collinear points, then

$$\frac{AP}{AB.AC.AD} + \frac{BP}{BC.BD.BA} + \frac{CP}{CD.CA.CB} + \frac{DP}{DA.DB.DC} = 0.$$

In the first identity take Q at infinity, then since

$$BQ/AQ = 1, \quad CQ/AQ = 1, \quad DQ/AQ = 1,$$

the required result follows.

Ex. 1. *Show that the first result is true for n points A, B, ... and $(n-2)$ points P, Q, ...*

Ex. 2. *Show that the second result is true for n points A, B, ... and $(n-1)$ points P, Q, ...*

Ex. 3. *Show that the third result is true for n points A, B, ... and $(n-2-m)$ points P, Q, ..., where m may be* 0, 1, 2, 3, ... $(n-2)$.

Ex. 4. *Enunciate the theorems obtained from Ex.* 2 *and Ex.* 3 *by taking the points P, Q, ... all coincident; and show that the theorems still hold when P is outside the line, provided the index of AP is even.*

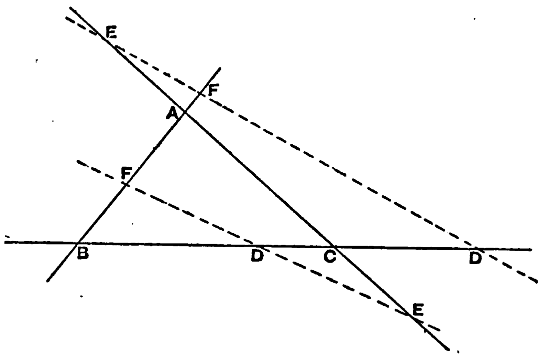
Use $AP^2 = Ap^2 + pP^2$, and the Binomial Theorem.

Menelaus's Theorem.

6. *If any transversal meet the sides BC, CA, AB of a triangle in D, E, F, then*

$$AF \cdot BD \cdot CE = -FB \cdot DC \cdot EA.$$

The transversal must cut all the sides externally, or two sides internally and one externally; for as a point proceeds along the transversal from infinity, at a point where the transversal cuts a side internally, the point enters the



triangle and at the point where the point leaves the triangle, the transversal must cut another side internally. Hence of the ratios $AF:FB$, $BD:DC$, $CE:EA$, one is negative and the other two are either both positive or both negative. Hence the sign of the formula is correct.

To prove that the formula is numerically correct, drop the perpendiculars p, q, r from A, B, C on the transversal. Then $AF/FB = p/q$, and $BD/DC = q/r$, and $CE/EA = r/p$.

Hence, multiplying, we see that the formula is true numerically.

Conversely, *if three points D, E, F, taken on the sides BC, CA, AB of a triangle, satisfy the relation*

$$AF \cdot BD \cdot CE = -FB \cdot DC \cdot EA,$$

then D, E, F are collinear.

For, if not, let DE cut AB in F'. Then since D, E, F' are collinear, we have

$$AF' . BD . CE = -F'B . DC . EA.$$

But by hypothesis we have

$$AF . BD . CE = -FB . DC . EA.$$

Dividing, we get $AF' : F'B :: AF : FB$; hence

$$AF' + F'B : AF + FB :: AF' : AF,$$

i. e. $AF' = AF$, i. e. F' coincides with F. Hence D, E, F are collinear.

Ex. 1. *Show that the above relation is equivalent to*

$$\sin ACF . \sin BAD . \sin CBE$$
$$= -\sin FCB . \sin DAC . \sin EBA.$$

For $AF : FB = \Delta ACF : \Delta FCB$
$$= \tfrac{1}{2} AC . CF \sin ACF : \tfrac{1}{2} FC . CB \sin FCB.$$

Ex. 2. *If any transversal cut the sides $AB, BC, CD, DE, \ldots$ of any polygon in the points $a, b, c, d, \ldots$, show that the continued product of the ratios*

$$Aa/Ba,\ Bb/Cb,\ Cc/Dc,\ Dd/Ed,\ \ldots \textit{ is unity.}$$

Let AC cut the transversal in γ, AD in δ, and so on,

then $Aa/Ba \times Bb/Cb \times C\gamma/A\gamma = 1$

and $A\gamma/C\gamma \times Cc/Dc \times D\delta/A\delta = 1$, and so on.

Multiplying up and cancelling, we get the theorem.

Ex. 3. *If on the four lines AB, BC, CD, DA there be taken four points a, b, c, d such that*

$$Aa . Bb . Cc . Dd = aB . bC . cD . dA,$$

show that ab and cd meet on AC and ad and bc meet on BD.

Apply Menelaus's Theorem to ABD and ad and to BCD and bc; multiply, and divide by the given relation; and we see that ad and bc meet BD in the same point; similarly for AC.

Ex. 4. *If the sides of the triangle ABC which is inscribed in a circle be cut by any transversal in D, E, F, show that the product of the tangents from D, E, F to the circle is numerically equal to $AF . BD . CE$.*

Ex. 5. *Construct geometrically the ratio $a/b \div c/d$.*

Divide AB in the ratio a/b and AC in the ratio c/d.

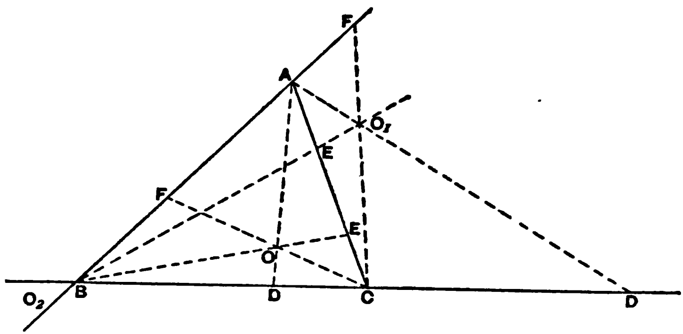
Ex. 6. *The bisectors of two angles of a triangle and the bisector of the supplement of the third angle meet the opposite sides in collinear points.*

Ceva's Theorem.

7. *If the lines joining any point to the vertices A, B, C of a triangle meet the opposite sides in D, E, F, then*

$$AF \,.\, BD \,.\, CE = FB \,.\, DC \,.\, EA.$$

To verify the sign of the formula. O the point in which AD, BE, CF meet must be *either* inside the triangle, in which case each of the ratios $AF : FB$ and $BD : DC$ and $CE : EA$ is positive, *or* as at O_1 or O_2, in which cases two of the ratios



are negative and one positive. Hence the sign of the formula is correct.

To prove the formula numerically, we have

$$\begin{aligned} AF : FB &:: \Delta\, ACF : \Delta\, FCB :: \Delta\, AOF : \Delta\, FOB \\ &:: \Delta\, ACF - \Delta\, AOF : \Delta\, FCB - \Delta\, FOB \\ &:: \Delta\, AOC : \Delta\, BOC. \end{aligned}$$

Similarly $BD : DC :: \Delta\, BOA : \Delta\, AOC$

and $CE : EA :: \Delta\, COB : \Delta\, AOB.$

Hence, multiplying, we see that the formula is true numerically.

Conversely, *if three points D, E, F, taken on the sides BC, CA, AB of a triangle, satisfy the relation*

$$AF \,.\, BD \,.\, CE = FB \,.\, DC \,.\, EA,$$

then AD, BE, CF are concurrent.

For, if not, let AD, BE cut in O; and let CO cut AB in F'. Then since AD, BE, CF' are concurrent, we have

$$AF' . BD . CE = F'B . DC . EA.$$

But by hypothesis we have

$$AF . BD . CE = FB . DC . EA.$$

Dividing, we get $AF' : F'B :: AF : FB$. Hence F and F' coincide, i.e. AD, BE, CF are concurrent.

Ex. 1. *In the figure, show that*

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1.$$

Ex. 2. *Show that the necessary and sufficient condition that Aa, Bb, Cc should meet in a point is*

$$\sin aAB . \sin bBC . \sin cCA = \sin CAa . \sin ABb . \sin BCc.$$

Ex. 3. *If the lines Aa, Bb, Cc, Dd, ... drawn through the vertices of a plane polygon $ABCD$ ... in the same plane meet in a point, then the continued product of such ratios as* $\sin aAB : \sin ABb$ *is unity.*

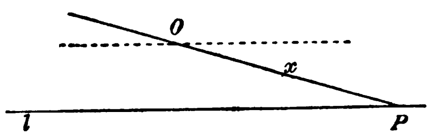
Ex. 4. *If the lines joining a fixed point O to the opposite vertices of a polygon of an odd number of sides meet the sides AB, BC, CD, DE,... in the points a, b, c, d,..., show that the continued product of such ratios as Aa/aB is unity.*

For $Aa/aB = AO . aO \sin AOa / aO . BO \sin aOB$.

Ex. 5. *A circle meets BC in D, D', CA in E, E', and AB in F, F'. Show that if AD, BE, CF meet in a point, so do AD', BE', CF'.*

The straight line at infinity.

8. Take any fixed point O and a fixed line l. Then any line x through O cuts l in a point P. Now rotate x about O

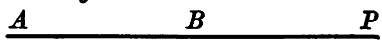


so that x may become more and more nearly parallel to l. Then P recedes indefinitely along l; and in the limit when

x is parallel to l, P is said to be the point at infinity upon l. Hence *two parallel lines intersect in a point at infinity.* Notice that the two points at infinity on a line are coincident; for instance in the figure each point is the limit of P when x is parallel to l.

Now draw a plane β parallel to any plane a. This will be parallel to every line in the plane a. Hence the point at infinity on each line in a lies on the plane β. Now every two planes intersect in a line. Hence all the points at infinity in the plane a lie on a single straight line (called *the straight line at infinity* in this plane), viz. the line of intersection of the planes a and β.

9. *The point at infinity on the line of any segment divides this segment externally in the ratio* -1.



For let P divide the segment AB externally. Then

$$\frac{AP}{PB}=-\frac{AP}{BP}=-\frac{AB+BP}{BP}=-\frac{AB}{BP}-1.$$

Also when P is at infinity, the limit of AB/BP is zero; for AB is finite and BP is infinite.

We may say that *a segment is bisected externally by the point at infinity on its line.*

I

1. If A, B, C are the angles of a triangle and A', B', C' the angles which the sides BC, CA, AB make with any line, then $\sin A \,.\, \sin A' + \sin B \,.\, \sin B' + \sin C \,.\, \sin C' = 0$.

2. OL, OM, ON are any three lines through O and PL, PM, PN make equal angles with OL, OM, ON in the same way; show that

$$PL \,.\, \sin MON + PM \,.\, \sin NOL + PN \,.\, \sin LOM = 0.$$

3. If A, B, C are the angles of a triangle and A', B', C' the angles which the sides BC, CA, AB make with any line, then $\dfrac{\sin B' \,.\, \sin C'}{\sin B \,.\, \sin C}+\dfrac{\sin C' \,.\, \sin A'}{\sin C \,.\, \sin A}+\dfrac{\sin A' \,.\, \sin B'}{\sin A \,.\, \sin B}=-1.$

4. If OA, OB, OC are any three lines through O, and PA, PB, PC are the three perpendiculars from any point P to OA, OB, OC, then
$\Sigma\{PB \,.\, PC \sin BOC\} = -PO^2 \sin BOC \,.\, \sin COA \,.\, \sin AOB.$

5. Through the vertices A, B, C of a triangle are drawn the parallels AX, BY, CZ to meet the sides BC, CA, AB at X, Y, Z; show that

$$\frac{BX \,.\, CX}{AX^2} + \frac{CY \,.\, AY}{BY^2} + \frac{AZ \,.\, BZ}{CZ^2} = 1.$$

6. If VA, VB, VC, VD, VO, VO' are any six concurrent lines, and if $a = \sin OVA \div \sin O'VA$, and so on, show that

$$\frac{\sin AVC}{\sin AVD} \div \frac{\sin BVC}{\sin BVD} = \frac{\gamma - a}{\delta - a} \div \frac{\gamma - \beta}{\delta - \beta}.$$

7. Three lines OAA', OBB', OCC' are cut by two lines ABC, $A'B'C'$; show that

$$\frac{OA}{OC} \div \frac{AB}{BC} = \frac{OA'}{OC'} \div \frac{A'B'}{B'C'}$$

and $$\frac{AA' \,.\, BC}{OA'} + \frac{BB' \,.\, CA}{OB'} + \frac{CC' \,.\, AB}{OC'} = 0.$$

8. If the polygon $A'B'C'D'\ldots$ is inscribed in the polygon $ABCD\ldots$, so that A' is on AB, B' on BC, and so on, and O is any point in the plane, show that the continued product of such ratios as $\sin AOA'/\sin A'OB \div AA'/A'B$ is unity.

9. If n points A, B, C, ... and n' points A', B', C', ... are collinear, and if G and G' are points on the line AB ... such that $GA + GB + GC + \ldots = 0$ and $G'A' + G'B' + G'C' + \ldots = 0$, show that $n \,.\, n' \,.\, GG' = \Sigma(AA' + AB' + AC' + \ldots)$.

10. If A, B, X, Y are collinear points such that $AX \,.\, AY = BX \,.\, BY$, then AB and XY have the same bisector unless A and B or X and Y coincide.

11. A line meets BC, CA, AB at D, E, F. P, Q, R bisect EF, FD, DE. AP, BQ, CR meet BC, CA, AB at X, Y, Z. Show that X, Y, Z are collinear.

12. A transversal meets the sides of a polygon $ABCD\ldots$ at A', B', C', ... and meets any lines through the vertices A, B, C, ... at A'', B'', C'', ...; show that the continued product of such ratios as $\sin A'BB''/\sin B''BB' \div A'B''/B''B'$ is unity.

13. If the lines AB, BC, CD, DA, which are not in the same plane, be met by any plane at A', B', C', D', then

$$AA' \cdot BB' \cdot CC' \cdot DD' = A'B \cdot B'C \cdot C'D \cdot D'A.$$

14. AO meets BC at D, BO meets CA at E, CO meets AB at F. GH is equal and parallel to BC and passes through A. BC meets GO at L and HO at K. Similarly segments such as KL are constructed on CA and AB. Show that the product of these segments is $AF \cdot BD \cdot CE$.

15. AO meets BC at P, BO meets CA at Q, CO meets AB at R. PU meets QR at X, QU meets RP at Y, RU meets PQ at Z. Show that AX, BY, CZ are concurrent.

16. Through the vertices of a triangle are drawn parallels to the reflexions of the opposite sides in any line. Show that these parallels are concurrent.

17. On the sides AB, AC of a triangle are taken the points X, Y, and in BC a point P is taken such that $AB \cdot AY \cdot PC = AC \cdot AX \cdot BP$. Show that AP will bisect XY.

CHAPTER II

HARMONIC RANGES AND PENCILS

1. *A range* or *row* is a set of points on the same line, called the *axis* or *base* of the range.

A *pencil* is a set of lines, called *rays*, passing through the same point, called the *vertex* or *centre* of the pencil.

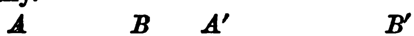
If A, B, A', B' are collinear points such that

$$AB : BA' :: AB' : A'B'$$

or (which is the same thing) such that

$$AB/BA' = -AB'/B'A',$$

then $(ABA'B')$ is called a *harmonic range.* A, A' and B, B' are called *harmonic pairs* of points; and A, A' are said to be *conjugate*, as also B, B'. Also A is said to be the *fourth harmonic* of A' (and A' of A) for B and B'; so B is said to be the fourth harmonic of B' (and B' of B) for A and A'. Also AA' and BB' are called *harmonic segments* and are said to divide one another harmonically. The briefest and clearest way of stating the harmonic relation is to say that (AA', BB') is harmonic. The relation may be stated in words thus—each pair of harmonic points divides the segment joining the other pair in the same ratio internally and externally.



For $$BA : AB' = -BA' : A'B'.$$

Ex. 1. *The centres of similitude of two circles divide the segment joining the centres of the circles harmonically.*

Ex. 2. (BC, XX'), (CA, YY'), (AB, ZZ') *are harmonic ranges; show that if* AX, BY, CZ *are concurrent, then* $X'Y'Z'$

are collinear, and that if $X'Y'Z'$ are collinear, then AX, BY, CZ are concurrent.

Use the theorems of Ceva and Menelaus.

2. *If (AA', BB') be harmonic, then*

$$\frac{2}{AA'} = \frac{1}{AB} + \frac{1}{AB'}, \quad \frac{2}{BB'} = \frac{1}{BA} + \frac{1}{BA'},$$

$$\frac{2}{A'A} = \frac{1}{A'B} + \frac{1}{A'B'}, \quad \frac{2}{B'B} = \frac{1}{B'A} + \frac{1}{B'A'}.$$

Taking any one of these formulae, say

$$\frac{2}{A'A} = \frac{1}{A'B} + \frac{1}{A'B'},$$

choose A' as origin in the defining relation

$$AB : BA' :: AB' : A'B'$$

and use abridged notation. Then $AB \,.\, A'B' = BA' \,.\, AB'$ gives us

$$(b-a)\,b' = (-b)\,(b'-a) \text{ or } bb' - ab' + bb' - ab = 0,$$

$$\text{or } 2bb' = ab + ab' \text{ or } \frac{2}{a} = \frac{1}{b} + \frac{1}{b'}.$$

Conversely, if any one of these relations is true, then (AA', BB') is harmonic.

For, retracing our steps, we see that

$$AB : BA' :: AB' : A'B'.$$

Notice that the formulae of this article are algebraical. To obtain numerical formulae we must pay attention to the signs. For instance (with the figure of § 1) the algebraical formula $\dfrac{2}{A'A} = \dfrac{1}{A'B} + \dfrac{1}{A'B'}$ gives the numerical formula $\dfrac{2}{A'A} = \dfrac{1}{A'B} - \dfrac{1}{A'B'}$ since $A'B'$ has the opposite direction to $A'A$ and $A'B$.

The proposition of this article may be enunciated thus. *The geometric mean between two lengths is equal to the geometric mean between the arithmetic mean and the harmonic mean.*

For $\dfrac{2}{AA'} = \dfrac{1}{AB} + \dfrac{1}{AB'}$ gives us

$$AB \,.\, AB' = \tfrac{1}{2}(AB + AB')\,AA'$$

Now $\frac{1}{2}(AB + AB')$ is the arithmetic mean, and AA' is

defined to be the harmonic mean, between AB and AB'; also $AB \,.\, AB'$ is the square of the geometric mean.

The formula $AB \,.\, AB' = A\beta \,.\, AA'$ is sometimes useful.

Ex. *If (AA', BB') be harmonic, and P be any point on the line AB', show that*

$$2 \,.\, \frac{PA'}{AA'} = \frac{PB}{AB} + \frac{PB'}{AB'}.$$

Put $PA' = PA + AA'$, &c.

3. *If (AA', BB') be harmonic, then $aA^2 = aB \,.\, aB'$; and conversely, if $aA^2 = aB \,.\, aB'$, then (AA', BB') is harmonic, a being the middle point of AA'.*

For taking a as the origin in the defining relation

$$AB : BA' :: AB' : A'B',$$

we have $$(b - a)(b' - a') = (a' - b)(b' - a).$$

But $a' = -a$, hence $(b - a)(b' + a) = (-a - b)(b' - a)$,

i.e. $bb' + \cancel{ba} - ab' - a^2 + ab' - a^2 + bb' - \cancel{ba} = 0$,

i.e. $bb' = a^2$, i.e. $aA^2 = aB \,.\, aB'$.

The converse follows by retracing our steps.

Ex. 1. *Show that the middle point of either of two harmonic segments is outside the other segment.*

Ex. 2. *One and only one segment XY can be found to divide two given collinear segments AB and CD harmonically.*

Take any point P not on the given line. Through ABP and CDP construct circles cutting again in Q. Let PQ cut $ABCD$ in O. From O draw tangents to the circles. With O as centre and any one of these tangents as radius, describe a circle. This circle will cut the given line in the required points X and Y. For

$$OX^2 = OY^2 = OP \,.\, OQ = OA \,.\, OB = OC \,.\, OD.$$

Also no other such segment can be found. For let O' bisect such a segment $X'Y'$. Then

$$O'A \,.\, O'B = O'X'^2 = O'C \,.\, O'D.$$

Hence O' is on the radical axis of the above two circles and therefore coincides with O; and then X' and Y' coincide with X and Y.

The points X, Y may be real, coincident or imaginary (see III. 1).

4. *To find the relation between four harmonic points and a fifth point on the same line.*

Let (AA', BB') be harmonic, and take the fifth point P as origin. Then by definition $AB/BA' = -AB'/B'A'$.

But $AB = PB - PA = b - a$, &c. Hence

$$(b-a)(a'-b') + (a'-b)(b'-a) = 0$$

$$\text{or } 2aa' + 2bb' = (a+a')(b+b'),$$

i.e. $2.PA.PA' + 2.PB.PB' = (PA + PA')(PB + PB')$.

Conversely, *if this relation hold, (AA', BB') is harmonic.* For reasoning backwards we deduce the relation

$$AB/BA' = -AB'/B'A'.$$

If (AA', BB') be harmonic, and a bisect AA' and β bisect BB', then $PA.PA' + PB.PB' = 2.Pa.P\beta$.

For $PA + PA' = 2.Pa$ and $PB + PB' = 2.P\beta$.

The best way of verifying any relation connecting four harmonic points is to take one of the points as origin. The relation must then reduce to one of those in § 2.

Ex. 1. $2.AB'.BA' = AA'.BB'$.

Ex. 2. $A'A^2 + B'B^2 = 4.a\beta^2$.

Ex. 3. $PA.A'B' + PA'.AB + PB.B'A + PB'.BA' = 0$.

Ex. 4. $BA : BA' :: \beta B : A'\beta$.

Ex. 5. $PA.PA' - PB^2 + 2.aB.P\beta = 0$.

5. If B, B' divide AA' in the same ratio internally and externally, then by definition (AA', BB') is a harmonic range. Now suppose this ratio is one of equality, then B becomes the internal bisector of the segment AA', i.e. B is the middle point of AA'; also B' becomes the external bisector of the segment AA', i.e. a point such that $AB' = A'B'$, B' being outside AA'. Hence (see I. 9) B' is the point at infinity on AA'. Hence the theorem—

Every segment is divided harmonically by its middle point and the point at infinity on the line, or, in other words, by its internal and external bisectors.

6. *If any two points of a harmonic range coincide, then a third point coincides with them and the fourth may be anywhere on the line.*

Suppose AA' coincide. Then B lying between A and A' must coincide with them. So for BB'.

Suppose AB coincide. Then $AB = 0$; hence, from the defining relation $AB \,.\, A'B' = BA' \,.\, AB'$, we conclude that $BA' = 0$ or $AB' = 0$, i.e. ABA' coincide or ABB'. So for AB', $A'B$, $A'B'$.

Again, if ABA' coincide, then $AB = 0$ and $BA' = 0$; hence the relation $AB \,.\, A'B' = BA' \,.\, AB'$ is satisfied wherever B' is. So for $BA'B'$, &c.

7. A pencil of four concurrent rays is called a *harmonic pencil* if every transversal cuts it in a harmonic range.

Harmonic pencils exist for—

If a pencil be obtained by joining any point to the points of a harmonic range, then every transversal cuts this pencil in a harmonic range.

Let (AA', BB') be a harmonic range and V any point. Join V to $AA'BB'$, and let any transversal cut the joining lines in $aa'bb'$.

Then $$ab : ba' = \Delta\, aVb : \Delta\, bVa'$$
$$= Va \,.\, Vb \sin aVb : Vb \,.\, Va' \sin bVa'.$$

Hence $$\frac{ab}{ba'} \div \frac{ab'}{b'a'} = \frac{\sin aVb}{\sin bVa'} \div \frac{\sin aVb'}{\sin b'Va'}.$$

Now $aVb' = AVB'$; but for the transversal $a\beta'$ we should have $$aV\beta' = 180° - AVB'.$$

So in all cases aVb' is either equal to or supplemental to AVB'; hence in all cases $\sin aVb' = \sin AVB'$. So for the other angles.

Hence $$\frac{ab}{ba'} \div \frac{ab'}{b'a'} = \frac{\sin AVB}{\sin BVA'} \div \frac{\sin AVB'}{\sin B'VA'}$$
$$= \frac{AB}{BA'} \div \frac{AB'}{B'A'} \text{ by similar reasoning}$$
$$= -1 \text{ by definition.}$$

Hence $ab/ba' \div ab'/b'a' = -1$; hence (aa', bb') is a harmonic range.

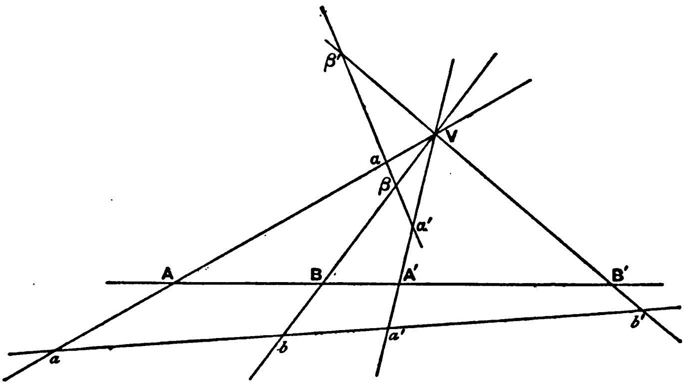
We denote the pencil subtended by $ABA'B'$ at V by $V(ABA'B')$; and we may briefly state the above theorem

thus—if (AA', BB') is a harmonic range, then $V(AA', BB')$ is a harmonic pencil (or more briefly still—is harmonic).

We may enunciate the above theorem also in the following way. *The projection of a harmonic range is a harmonic range*; for $(aba'b')$ is the projection of $(ABA'B')$ from the vertex V. (See Chapter IV.)

There are two important particular cases.

(1) The vertex V may be at infinity. But the proof holds



however distant V is, and therefore holds in the limit when V is at infinity. Hence *the parallel projection* (*and in particular—the orthogonal projection*) *of a harmonic range is a harmonic range.*

(2) The section may be parallel to one of the rays of the pencil; for instance b' may be at infinity. As before the proof still holds, but now b bisects aa' since (aa', bb') is harmonic and b' is at infinity. Hence we have the theorem. *If a section of the harmonic pencil $V(AA', BB')$ be drawn parallel to VB' cutting VA, VB, VA' at a, b, a', then $ab = ba'$; and conversely if $ab = ba'$, the pencil $V(aa', bb')$ is harmonic if Vb' is parallel to ab.*

Hence *a pencil formed by any two lines and the bisectors of the angles between them is harmonic.* For let VB, VB' bisect the angles between VA, VA'. Draw the section aba'

parallel to VB'. Then obviously $ab = ba'$; also b' is at infinity. Hence $V(aa', bb')$ i.e. $V(AA', BB')$ is harmonic.

Also *if two conjugate rays of a harmonic pencil are perpendicular, they bisect the angles between the other pair.* For with the same construction, we have $ab = ba'$ and the angles at b right angles; hence $aVb = bVa'$.

Ex. 1. *If $V(AA', BB')$ be harmonic, prove that*
$$2 \cot AVA' = \cot AVB + \cot AVB'.$$
Take a transversal perpendicular to VA.

Ex. 2. *Also if Va bisect the angle AVA', then*
$$\tan^2 aVA = \tan aVB . \tan aVB'.$$
Take a transversal perpendicular to Va.

Ex. 3. $$2 \sin AVB' . \sin BVA' = 2 \sin AVB . \sin A'VB' = \sin AVA' . \sin BVB'.$$
For $p . AB' = VA . VB' \sin AVB'$ and so on.

Ex. 4. $$2 . \frac{\sin PVA'}{\sin AVA'} = \frac{\sin PVB}{\sin AVB} + \frac{\sin PVB'}{\sin AVB'}$$
where VP is an arbitrary line through V.

Ex. 5. *Given two pairs of lines VA, VB and VC, VD through the same point, one pair of lines, and only one, can be found harmonic with these.*

This follows at once from Ex. 2 of § 3. The lines may be real, coincident or imaginary (see III. 1).

8. *The polar of a point O for two lines BA* and *BC* is defined to be the fourth harmonic of BO for BA and BC.

The pole of a line LM for two points A, B is defined to be the fourth harmonic of the meet of LM and AB for A and B.

If through O there be drawn the transversal OPQ cutting BA in P and BC in Q, then the locus of R, the fourth harmonic of O for P and Q, is the polar of O for BA and BC.

For the pencil $B(OPRQ)$ is harmonic.

If the two lines BA, BC be parallel, i.e. if B be at infinity, the above still holds, if we consider B to be the limit of a finite point.

To construct the polar of O for ΩA, ΩC where Ω is at infinity, draw any transversal OPQ meeting ΩA in P and ΩC in Q, and take R so that $(OPQR)$ is harmonic, and

through R draw a parallel ΩR to ΩA and ΩC; then ΩR is the polar of O for the parallels ΩA, ΩC.

Ex. 1. *The polars of any point for the three pairs of sides of a triangle meet the opposite sides in three collinear points.*

Let AO, BO, CO meet the opposite sides in P, Q, R, and let the polars of O meet these sides in P', Q', R'.

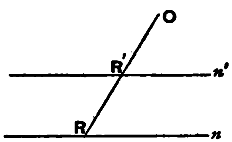
Then $BP/PC = -BP'/P'C$, and so on.

Now use the theorems of Menelaus and Ceva.

Ex. 2. *The poles of any line for the pairs of vertices of a triangle connect concurrently with the opposite vertices.*

***9.** *Through a given point O is drawn a line meeting two fixed lines in P and Q, and on OPQ is taken the point X such that $1/OX = 1/OP + 1/OQ$; find the locus of X.*

Take the polar n of O for the two given lines, and let OPQ meet this line in R. Then we know that



$$2/OR = 1/OP + 1/OQ.$$

Now draw parallel to n and half-way between O and n the line n' cutting OPQ in R'.

Then $OR' = OR/2$, i.e. $2/OR = 1/OR'$.

Hence $1/OR' = 1/OP + 1/OQ$; hence n' is the required locus.

Ex. 1. *A transversal through the fixed point O meets fixed lines in A, B, C, ... and on OA is taken a point P such that $1/OP = 1/OA + 1/OB + 1/OC + \dots$; find the locus of P.*

Replace $1/OA + 1/OB$ by $1/OL$, and so on.

Ex. 2. *A transversal through the fixed point O meets fixed lines in A, B, C, ...; find the direction of the transversal when $\Sigma 1/OA$ is* (i) *a maximum,* (ii) *a minimum.*

Perpendicular and parallel to the locus of P.

Ex. 3. *A transversal through the fixed point O meets fixed lines in A, B, C, ... and on OA is taken a point P such that $1/OP = a/OA + b/OB + c/OC + \dots$, where a, b, c, ... are any multipliers; find the locus of P. Also find the direction of the transversal when $\Sigma a/OA$ is* (i) *a max.,* (ii) *a min.*

Whatever a, b, c, ... are, we can, by taking the integer k large enough, make ka, kb, kc, ... all integers. Hence

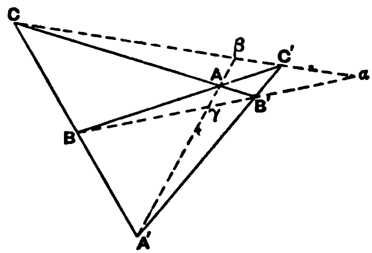
$$k/OP = a'/OA + b'/OB + c'/OC + \dots$$

where k, a', b', c', ... are all integers. Now by Ex. 1 find the locus of Q such that

$$1/OQ = (1/OA + \dots a' \text{ times}) + (1/OB + \dots b' \textit{ times}) + \dots$$

and draw a parallel through P to the locus of Q such that $OP = k \cdot OQ$. This parallel is the required locus.

10. A *complete quadrilateral* is formed by four lines called the *sides* which meet in six points called the *vertices* of the quadrilateral. These six points can be joined by three other lines called the *diagonals*. The diagonals are also called the *harmonic lines* of the quadrilateral and the harmonic lines form the sides of the *harmonic triangle*. These names are derived from the following property—called *the harmonic property of a complete quadrilateral.*



Each diagonal of a complete quadrilateral is divided harmonically by the other two diagonals.

Let the four sides of the complete quadrilateral meet in the three pairs of opposite vertices AA', BB', CC'. Then AA', BB', CC', or $\beta\gamma$, γa, $a\beta$ are the harmonic lines. We have to show that the ranges $(AA', \beta\gamma)$, $(BB', \gamma a)$, $(CC', a\beta)$ are harmonic.

To prove that $(AA', \beta\gamma)$ is harmonic, take γ' the fourth harmonic of β for AA'. Let $B\gamma'$ cut CC' at a'. Then the pencil $B(A'A, \gamma'\beta)$ is harmonic; hence its section $(CC', a'\beta)$ is harmonic; i.e. a' is the fourth harmonic of β for CC'. In exactly the same way we prove that $B'\gamma'$ cuts CC' in the fourth harmonic of β for CC'. Hence $B\gamma'$ and $B'\gamma'$ are the same line; i.e. BB' cuts AA' in the fourth harmonic of β for AA'; i.e. $(AA', \beta\gamma)$ is harmonic.

Similarly $(BB', \gamma a)$ and $(CC', a\beta)$ are harmonic.

For a proof by projection see Chapter IV.

11. *Using a ruler only, construct the fourth harmonic of a given point for two given points.*

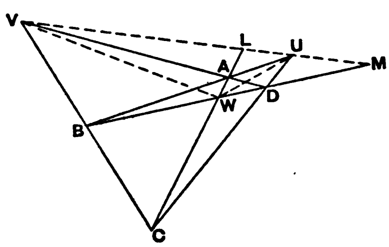
To construct the fourth harmonic of γ for B and B'. On any line through γ take two points A and A'. Let $A'B$, AB' cut in C and AB, $A'B'$ in C'. Then CC' cuts BB' in the required point a. For BB' is a diagonal of the complete quadrilateral formed by AB, AB', $A'B$, $A'B'$; hence $(BB', \gamma a)$ is harmonic.

Ex. 1. *AO, BO, CO meet BC, CA, AB in P, Q, R; QR, RP, PQ meet BC, CA, AB in X, Y, Z. Show that (BC, PX), (CA, QY), (AB, RZ) are harmonic ranges, and that XYZ are collinear.*

Consider the quadrilateral $AQORA$.

Ex. 2. *If a transversal meet BC, CA, AB in X, Y, Z, and the join of A to the meet of BY and CZ cut BC in P; show that (BC, PX) is harmonic, and that the three lines formed like AP are concurrent.*

12. A *complete quadrangle* is formed by four points called the *vertices* which are joined by six lines called the *sides* of the quadrangle. These six lines meet in three other points called *the harmonic points* of the quadrangle; and the harmonic points are the vertices of the *harmonic triangle.* Some writers give the name diagonal-points to the harmonic points.



The following is *the harmonic property of a complete quadrangle.*

The angle at each harmonic point is divided harmonically by the joins to the other harmonic points.

Let $ABCD$ be the four points forming the quadrangle. Then U, V, W are the harmonic points of the quadrangle; and we have to show that the pencils

$$U(AD, VW), \quad V(BA, WU), \quad W(CD, UV)$$

are harmonic.

To show that the pencil $W(CD, UV)$ is harmonic, it is sufficient to show that the range (LM, UV) is harmonic, L being the meet of AC and UV, and M of BD and UV. And this is true; for VU is a diagonal of the quadrilateral $ABCDA$.

Similarly $U(AD, VW)$ and $V(BA, WU)$ are harmonic.

For a proof by projection, see Chapter IV.

13. *Using a ruler only, construct the fourth harmonic of a given line for two given lines.*

To construct the fourth harmonic of VU for VA and VB. Through any point U on VU draw any two lines UAB and UDC, cutting VA in A and D, and VB in B and C. Then if AC and BD meet in W, VW is the required line. For U, V, W are the harmonic points of the quadrangle A, B, C, D. Hence $V(BA, WU)$ is harmonic.

Ex. 1. *Through one of the harmonic points of a complete quadrangle is drawn the line parallel to the join of the other two harmonic points; show that two of the segments cut off between opposite sides of the quadrangle are bisected at the harmonic point.*

One point of each harmonic range being at infinity.

Ex. 2. *Through V, one of the harmonic points of a quadrangle, is drawn a line parallel to one side and meeting the opposite side in P and the join of the other harmonic points in Q; show that $VP = PQ$.*

Ex. 3. *In the figure of the quadrilateral in* § 10, *show that $A\alpha$, $A'\alpha$, $B\beta$, $B'\beta$, $C\gamma$, $C'\gamma$ form the six sides of a quadrangle.*

We have to show that the six lines pass three by three through four points. Consider αA, $\beta B'$, $\gamma C'$. From the quadrilateral $\beta\gamma B'C'\beta$ it follows that the join of α to the meet of $\beta B'$ and $\gamma C'$ passes through A. Hence αA, $\beta B'$, $\gamma C'$ are concurrent. Similarly αA, βB, γC are concurrent, also $\alpha A'$, βB, $\gamma C'$, and also $\alpha A'$, $\beta B'$, γC.

II

1. If AD, BE, CF are the perpendiculars on BC, CA, AB and if (BC, DP), (CA, EQ) and (AB, FR) are harmonic, show that PQR is the radical axis of the circum-circle and the nine-points circle.

2. In a complete quadrangle, the sides of the harmonic triangle meet the sides of the quadrangle in six new points which are the vertices of a quadrilateral.

3. If (AB, CD) is harmonic and O bisects AB and O' bisects CD, show that $AC:BD::CO.CO':BO.BO'$.

4. If (AB, CD) is harmonic, prove that

$$2\,OR.OR' = OP.OP' + OQ.OQ'$$

where O is any point on AB and P, P', Q, Q', R, R' bisect BC, AD, CA, BD, AB, CD.

5. ABC is a triangle. AO, BO, CO meet BC, CA, AB at P, Q, R. PQ meets AB at W and QR meets BC at U. Show that AU, CW meet on BQ.

CHAPTER III

HARMONIC PROPERTIES OF A CIRCLE

1. *Every line meets a circle in two points, real, coincident or imaginary.*

For take any line l cutting a circle in the points A and B. Now move l parallel to itself away from the centre of the circle. Then A and B approach, and ultimately coincide when l touches the circle. But when l moves still further from the centre, the points A and B disappear; yet, for the sake of continuity, we say that they still exist, but are *imaginary*. (See also XXVII.)

2. *From every point can be drawn to a circle two tangents, real, coincident or imaginary.*

For take any point T outside the circle, and let TP and TQ be the tangents from T to the circle. Now let T approach the centre O of the circle along OT. Then TP and TQ approach, and ultimately coincide when T reaches the circumference. But when T moves still further towards O, the tangents TP and TQ disappear; yet, for the sake of continuity, we say that they still exist, but are *imaginary*. (See also XXVII.)

3. Two points which divide any diameter of a circle harmonically are said to be *inverse points* for this circle.

If O be the centre and r the radius of the circle, then inverse points B, B' must lie on the same radius of the circle and be such that $OB \cdot OB' = r^2$; for if B, B' divide the diameter AA' harmonically, since O bisects AA', we have $OB \cdot OB' = OA^2$.

Ex. 1. *The inverse of any point at infinity for a circle is the*

centre of the circle; and conversely, the inverse of the centre is any point at infinity.

Ex. 2. *If four points (AA', BB') be harmonic, so are the four inverse points (aa', bb') for any circle whose centre is on the line.*

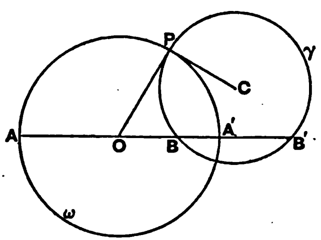
For $Oa = \frac{r^2}{OA}$, $Ob = \frac{r^2}{OB}$; *hence* $ab = -\frac{r^2 . AB}{OA . OB}$.

Ex. 3. *If BB' be a pair of inverse points on the diameter AA' of a circle, and if P be any point on the circle; then PA, PA' bisect the angle BPB', and the ratio $PB : PB'$ is independent of the position of P.*

4. Two *circles* are said to be *orthogonal* when the tangents to the circles at a point of intersection are at right angles.

Let the circles with centres A and B cut orthogonally at P. Draw the tangents PQ and PR at P. Then PA is perpendicular to PQ and so is PR. Hence PR coincides with PA, i.e. passes through A; so PQ passes through B. Hence $PA^2 + PB^2 = AB^2$ or briefly $a^2 + b^2 = \delta^2$. The two circles are also orthogonal at the other intersection; for both circles are symmetrical about the line AB. Notice that we have proved that *the tangent drawn from the centre of either circle to the other is equal to the radius of the first circle.*

5. *Every circle which passes through a pair of points inverse for a circle is orthogonal to this circle; and conversely, every circle orthogonal to a circle cuts every diameter of this circle in a pair of inverse points.*



First, let the circle γ pass through the inverse points BB' of the circle ω. Let P be one of the meets of ω and γ. Then

$$OB . OB' = OP^2.$$

Hence OP touches γ. Hence OPC is a right angle. Hence CP touches ω. Hence the tangents OP and CP are at right angles, i.e. the two circles are orthogonal.

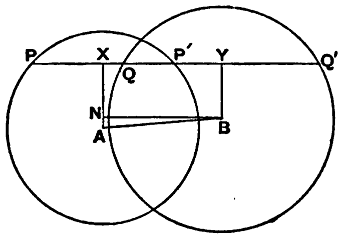
Second, let the two circles ω and γ be orthogonal.

Through the centre O of ω draw the diameter AA' cutting γ in BB'. Then since the circles are orthogonal, OPC is a right angle; hence OP touches γ. Hence $OB \,.\, OB' = OP^2$. Hence B and B' are inverse points for ω.

Ex. 1. *If a circle α divide one diameter of the circle β harmonically, it divides every diameter of β harmonically.*

Ex. 2. *On the diagonals of a complete quadrilateral as diameters are drawn three circles; show that each of these cuts orthogonally the circle about the harmonic triangle.*

6. *A line cuts two circles in the points PP' and QQ', so that (PP', QQ') is harmonic; show that the product of the perpendiculars from the centres of the circles on the line is constant.*



Let A be the centre and a the radius of one circle, and B and b those of the other circle. Let $AX = p$ and $BY = q$ be the perpendiculars from A and B on the line. Then X bisects PP', Y bisects QQ', and since (PP', QQ') is harmonic, we have

$$XP^2 = XQ \,.\, XQ'.$$

Draw BN perpendicular to AX. Denote AB by δ.

Now $$2pq = p^2 + q^2 - (p-q)^2 = a^2 - PX^2 + b^2 - QY^2 - AN^2$$
$$= a^2 + b^2 - PX^2 - QY^2 - \delta^2 + XY^2 = a^2 + b^2 - \delta^2.$$

For
$$XY^2 - PX^2 - QY^2 = (XY + QY)(XY - QY) - XP^2$$
$$= XQ' \,.\, XQ - XP^2 = 0.$$

Hence pq is constant.

Ex. 1. *If a line cut two orthogonal circles harmonically, it must pass through one of the centres.*

For $p = 0$ or $q = 0$.

Ex. 2. *If a line l cut one circle in the points PP' and another circle in the points QQ', which are such that (PP', QQ') is harmonic; show that the envelope of l is a conic whose foci are*

the centres of the circles. Show also that if the circles meet in C and D, the envelope touches the four tangents of the circles at C and D.

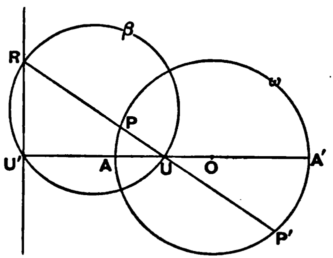
Since pq is constant, the first follows by Geometrical Conics. Also if l become the tangent at C, then PP' and Q coincide at C; hence (PP', QQ') is harmonic whatever point Q' is.

This envelope is called *the harmonic envelope* of the two circles.

Ex. 3. *The locus of the middle points of PP' and QQ' is the coaxal circle whose centre bisects AB.*

For the locus of X and Y is the auxiliary circle of the conic. Also each meet of the circles is on the locus; for when P and P' are at C, so is X.

7. *Through a point U is drawn a variable chord PP' of a circle and on PP' is taken the point R such that (UR, PP') is harmonic; to show that the locus of R is a line.*

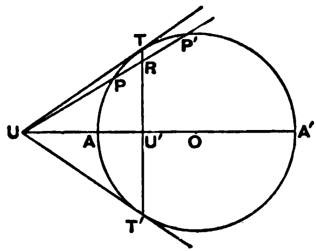


Take O the centre of the given circle ω. Let OU cut ω in AA'. From any position of R drop a perpendicular RU' to UO. On RU as diameter describe the circle β passing through U'. Now since (RU, PP') is harmonic, PP' are inverse points for β. Hence ω and β are orthogonal. Hence UU' are inverse for ω. Hence U' is a fixed point. Hence the locus of R is a fixed line, viz. the perpendicular to OU through the inverse of U for the given circle.

The locus of R is called the *polar* of U for the circle. We may briefly define the polar of a point for a circle as the locus of the fourth harmonics of the point for the circle. Also if RU' is given, U is called its *pole* for the circle, and U and RU' are said to be *pole and polar* for the circle.

8. *If U be outside the circle, the polar of U for the circle is the chord of contact of tangents from U to the circle.*

For take the chord UPP' very near the tangent UT. Then when PP' coincide, R, being between them, coincides with them; i.e. one position of R is at T. So another position of R is at T'. Hence TT' is the polar.



The polar of the centre of the circle is the line at infinity.

For if U coincide with O, then PP' is bisected at U. Hence R is at infinity.

The pole of the line at infinity for a circle is the centre of the circle.

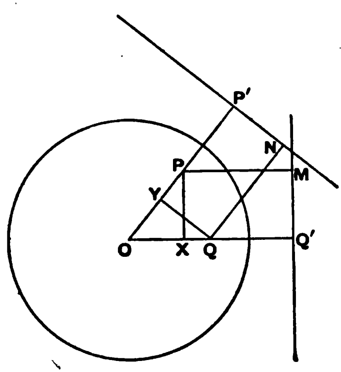
For if R be always at infinity, PP' is always bisected at U, i.e. U is the centre of the circle.

The polar of a point on the circle is the tangent at the point.

For suppose U to approach A, then since $OU \cdot OU' = OA^2$, we see that U' also approaches A. Hence when U is at A, U' is at A; and the polar of U, being the perpendicular to OU through U', is the tangent at U.

Similarly, *the pole of a tangent to a circle is the point of contact.*

9. Salmon's theorem.—*If P and Q be any two points and if PM be the perpendicular from P on the polar of Q for any circle, and if QN be the perpendicular from Q on the polar of P for the same circle, then $OP/PM = OQ/QN$, O being the centre of the circle.*



From P drop PX perpendicular to OQ and from Q drop

QY perpendicular to OP. Then P' being the inverse point of P, and Q' the inverse point of Q, we have

$$OP \cdot OP' = OQ \cdot OQ'.$$

Also since the angles at X and Y are right, we have

$$OY \cdot OP = OX \cdot OQ,$$

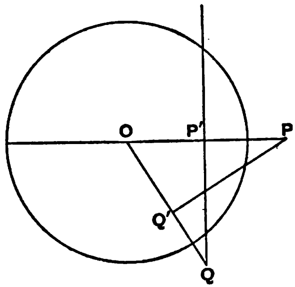
$$\therefore\ OP/OQ = OQ'/OP' = OX/OY = (OQ' - OX) \div (OP' - OY)$$
$$= XQ'/YP' = PM/QN.$$

Hence $$OP/PM = OQ/QN.$$

We may enunciate this theorem more briefly thus—*If p, q be the polars of P, Q for a circle whose centre is O, then*

$$OP/(P, q) = OQ/(Q, p).$$

10. *If the polar of P pass through Q, then the polar of Q passes through P.*



If the polar of P pass through Q, then, P' being the inverse of P, $P'Q$ is perpendicular to OP. Take Q' the inverse of Q. Then

$$OP \cdot OP' = OQ \cdot OQ'.$$

Hence $PP'QQ'$ are concyclic. Hence $OQ'P = OP'Q$ is a right angle. Hence PQ' is the polar of Q, i.e. the polar of Q passes through P.

The points P and Q are called *conjugate points* for the circle. We may define two conjugate points for a circle to be such that the polar of each for the circle passes through the other.

Note that if PQ cut the circle in real points RR', then, since the polar of P passes through Q, we see that (PQ, RR') is harmonic; and hence the polar of Q passes through P.

The pole of the join of P and Q is the meet of the polars of P and Q.

For if the polars of P and Q meet in R, then, since the polars of P and Q pass through R, therefore the polar of R passes through P and Q.

11. *On every line there is an infinite number of pairs of conjugate points for a given circle; and each of these pairs is harmonic with the pair of points in which the line meets the circle.*

On the line take any point P, and let the polar of P meet the line in P'. Then P and P' are conjugate points; for the polar of P passes through P'. Also if PP' meet the circle in RR', then (PP', RR') is harmonic; for P' is on the polar of P.

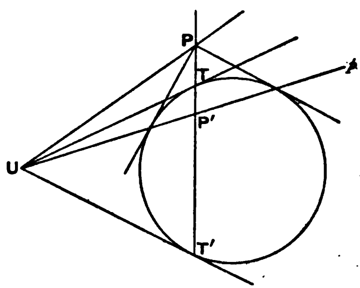
Conversely, *every two points which are harmonic with a pair of points on a circle are conjugate for the circle.*

12. *If the line p contain the pole of the line q, then q contains the pole of p.*

Let P be the pole of p and Q of q. We are given that p contains Q, i.e. that the polar of P passes through Q. Hence the polar of Q passes through P, i.e. q passes through P, i.e. q contains the pole of p.

The lines p and q are called *conjugate lines* for the circle. We may define two conjugate lines for a circle to be such that each contains the pole of the other.

Through every point can be drawn an infinite number of pairs of lines which are conjugate for the circle, and each of these is harmonic with the pair of tangents from the point.



For take any line p through the given point U and join U to the pole P of p. Then p and UP are conjugate lines, for UP contains the pole of p.

Draw the tangents UT and UT' from U, and let the polar TT' of U meet p in P'. TT' meets UP in P since U is on the polar of P. Now the range (PP', TT') is harmonic, for P' is on the polar of P; hence the pencil $U(PP', TT')$ is harmonic,

i.e. the conjugate lines p and UP are harmonic with the tangents from U.

Conversely, *every pair of lines which are harmonic with the pair of tangents from a point to a circle are conjugate for the circle.*

For let UQ and UQ' be harmonic with the tangents UT, UT' from U. Let UQ and UQ' cut the polar TT' of U in P and P'. Since $U(QQ', TT')$ is harmonic, hence (PP', TT') is harmonic. Hence UP' is the polar of P; for the polar of P passes through P' since (PP', TT') is harmonic, and passes through U since P is on the polar of U. Hence since the pole of UP' lies on UP, we see that UP and UP' are conjugate lines.

Ex. 1. *A point can be found conjugate to each of two given points; and a line can be found conjugate to each of two given lines.*

Ex. 2. *If the circle α be orthogonal to the circle β, then the ends of any diameter of α are conjugate for β.*

Ex. 3. *If $B'C'$ be the polar of A, $C'A'$ of B and $A'B'$ of C; then BC is the polar of A', CA of B' and AB of C'.*

Ex. 4. *M and N are the projections of a point P on a circle on two perpendicular diameters, Q is the pole of MN for the circle, and U and V are the projections of Q on the diameters. Show that UV touches the circle.*

UV is the polar of P.

Ex. 5. *If P and Q are conjugate points for a circle, then the circle on PQ as diameter is orthogonal to the given circle.*

For (see the figure of § 10) the circle on PQ as diameter passes through Q' which is the point inverse to Q; i.e. it passes through a pair of points which are inverse for the given circle.

Ex. 6. *If P, Q are conjugate points for a circle, and C bisects PQ, then the tangent from C to the given circle is equal to half PQ.*

13. Pairs of conjugate lines at the centre of a circle are called pairs of *conjugate diameters* of the circle.

Every pair of conjugate diameters of a circle is orthogonal.

Take any diameter AA' of a circle whose centre is O. The diameter conjugate to AA' is the line through O conjugate

to AA', i.e. is the join of O to the pole of AA'. But the tangents at A and A' meet at infinity in Ω, say. Hence $O\Omega$ is the conjugate diameter; hence the diameter conjugate to AA' is parallel to the tangent at A, i.e. is perpendicular to AA'.

Ex. 1. *The pole of a diameter is the point at infinity on any line perpendicular to the diameter; and the polar of any point Ω at infinity is the diameter perpendicular to any line through Ω.*

Ex. 2. *Any two points at infinity which subtend a right angle at the centre are conjugate.*

14. A triangle is said to be *self-conjugate* for a circle when every two vertices and every two sides are conjugate for the circle.

Such a triangle is clearly such that each side is the polar of the opposite vertex. Hence the other names—self-reciprocal or self-polar.

Self-conjugate triangles exist.

For on the polar of any point A take any point B. Then the polar of B passes through A and meets the polar of A in C, say. Then ABC is a self-conjugate triangle. For BC is the polar of A, CA is the polar of B; hence C, the meet of BC and CA, is the pole of AB. Hence AB are conjugate points, and BC, AC are conjugate lines. So for other pairs.

Ex. *The triangle formed by the line at infinity and any two perpendicular diameters of a circle is self-conjugate for the circle.*

15. *There is only one circle for which a given triangle is self-conjugate; and this is real only when the triangle is obtuse-angled.*

Suppose the triangle ABC is self-conjugate for the circle whose centre is O. Then since A is the pole of BC, it follows that OA is perpendicular to BC; so OB is perpendicular to CA, and OC to AB. Hence O is the orthocentre of ABC. Let OA meet BC in A', OB meet CA in B' and OC meet AB in C'. Then the square of the radius of the circle must be equal to $OA \cdot OA'$ and to $OB \cdot OB'$ and to $OC \cdot OC'$; and this

is possible if O is the orthocentre, for then these products are equal.

Now describe a circle (called the *polar circle* of the triangle) with the orthocentre O as centre and with radius ρ, such that $\rho^2 = OA \cdot OA' = OB \cdot OB' = OC \cdot OC'$. Then the triangle ABC is self-conjugate for this circle. For BC, being drawn through the inverse point A' of A perpendicular to OA, is the polar of A; so for CA and AB.

Also this circle is imaginary if the triangle is acute-angled; for then O is inside the triangle and hence ρ^2 $(= OA \cdot OA')$ is negative.

Ex. 1. *Describe a circle to cut the three sides of a given triangle harmonically. When is this circle real?*

Ex. 2. *The circle on each of the diagonals of a quadrilateral as diameter is orthogonal to the polar circle of each of the four triangles formed by the sides of the quadrilateral.*

For in the figure of II. 10 since $B'C$ is the polar of A' for the polar circle of $A'B'C$, A and A' are conjugate for this circle. Hence the circle on AA' as diameter is orthogonal to this circle; and similarly to the other polar circles.

Ex. 3. Hence the two sets of circles are coaxal. Hence *the middle points of the three diagonals of a quadrilateral lie on a line; and the four orthocentres of the four triangles formed by the sides of a quadrilateral lie on a line; and these lines are perpendicular.*

Ex. 4. *The centre of the circle circumscribing the harmonic triangle of the quadrilateral is collinear with the four orthocentres.*

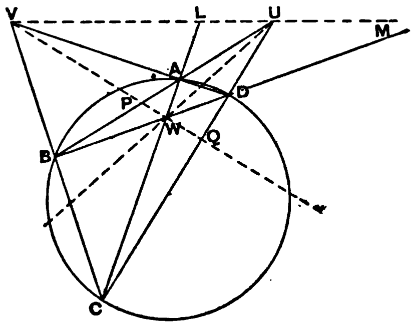
Ex. 5. *In any triangle the circles on the sides as diameters are orthogonal to the polar circle.*

Ex. 6. *If any point X be taken on the side BC of a triangle, the circle on AX as diameter is orthogonal to the polar circle of ABC.*

16. *The harmonic triangle of a quadrangle inscribed in a circle is self-conjugate for the circle.*

Let UVW be the harmonic triangle of the quadrangle $ABCD$ inscribed in a circle. Then UVW is self-conjugate for the circle.

Let UV meet AC in L and BD in M. Then since $V(WU, BA)$ is harmonic, hence (WL, AC) and (WM, BD)



are harmonic. Hence L and M lie on the polar of W, i.e. UV is the polar of W. Similarly VW is the polar of U, and WU of V.

17. *With the ruler only, to construct the polar of a given point for a given circle.*

To construct the polar of V for the given circle, draw through V any two chords AD and BC of the circle. Let BA, CD meet in U, and AC, BD meet in W. Then by the above theorem WU is the polar of V.

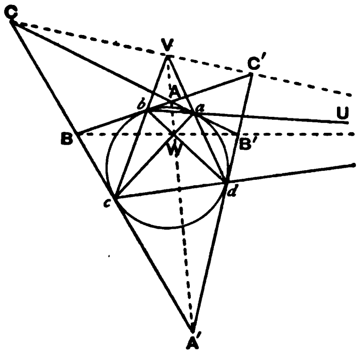
Ex. *Through U one of the harmonic points of a quadrangle inscribed in a circle is drawn a chord cutting the circle in aa', and the pairs of opposite sides in bb', cc'; show that if one of the segments aa', bb', cc' is bisected at U, the others are also bisected at U.*

Let the transversal cut the opposite side of the harmonic triangle in X, then UX divides each segment harmonically.

18. *The three diagonals of a quadrilateral circumscribing a circle form a triangle self-conjugate for the circle.*

Let the three diagonals AA', BB', CC' of the quadrilateral BA, AB', $B'A'$, $A'B$ circumscribing the circle form the triangle $\alpha\beta\gamma$. Then $\alpha\beta\gamma$ (the harmonic triangle of the quadrilateral) is self-conjugate for the circle.

Let AB', $B'A'$, $A'B$, BA touch the circle at a, d, c, b; and let UVW be the harmonic triangle of the quadrangle $abcd$. Then V is the pole of WU, since UVW is self conjugate by § 16. But V is the intersection of bc and ad, i. e. of the polars of B and B'. Hence V is the pole of BB'. Hence BB' coincides with WU, i. e. BB' passes through W and U. So AA' passes through W and V and CC' through V and U. Hence UVW coincides with $\alpha\beta\gamma$, which is therefore self-conjugate. Hence the harmonic triangle of a circumscribed quadrilateral is self-conjugate.



Notice that we have incidentally proved that *the harmonic triangle of a quadrilateral circumscribed to a circle coincides with the harmonic triangle of the inscribed quadrangle formed by the points of contact.*

Ex. 1. *The two lines joining those opposite intersections of common tangents of two circles which are not centres of similitude cut the line of centres in the limiting points of the two circles.*

Let these intersections be B, B' and C, C'. Then SS', BB', CC' are the diagonals of the quadrilateral of common tangents. Hence if BB', CC' cut SS' at L, L' and one another at Ω (at infinity), $LL'\Omega$ is self-conjugate for both circles. Hence L, L' are points on the line of centres which are conjugate for both circles; i.e. are the limiting points.

Ex. 2. *Show that the limiting points are harmonic with the centres of similitude.*

Ex. 3. *The lines joining the points of contact with one circle of the common tangents of two circles are either perpendicular to the line of centres or pass through a limiting point.*

19. *With the ruler only, to construct the pole of a given line for a given circle.*

This may be done by the above theorem; but better by finding by § 17 the meet of the polars of two points on the given line.

III

1. Given a pair of inverse points for a circle, the circle must be one of a certain system of coaxal circles.

2. If B, B' are inverse points on the diameter AA' of a circle whose centre is O, and if perpendiculars to AA' at A, A', B, B' meet any tangent to the circle at a, a', b, b', show that Oa and Oa' bisect the angle bOb' and that the ratio $Ob : Ob'$ is independent of the position of the tangent.

3. Through two given points draw a circle to cut a given segment harmonically.

4. R is any point on a given circle with centre O. A and B are fixed points collinear with O and equidistant from it. With A and B as centres and radii AR and BR are described circles. Show that the harmonic envelope of these circles is the same for all positions of R on the given circle.

5. If a, b, p are the polars of the points A, B, P for a circle whose centre is O, show that

$$\frac{(P, a)}{(P, b)} \div \frac{(A, p)}{(B, p)} = \frac{(O, a)}{(O, b)} = \frac{(B, a)}{(A, b)}.$$

6. P and Q are conjugate points for a circle. Show that the sum of the squares of the tangents from P and Q is equal to PQ^2. Also if U is the foot of the perpendicular from the centre of the circle on PQ, then the square of the tangent from U is equal to $PU \cdot UQ$.

7. In the figure of § 18 if AA' meets ab at P and cd at P', and so on, show that the six points P, P', Q, Q', R, R' lie three by three on four lines.

CHAPTER IV

PROJECTION

1. Given a figure ϕ in one plane π consisting of points A, B, C, ... and lines l, m, n, ..., we can construct another figure ϕ' consisting of *corresponding* points A', B', C', ... and lines l', m', n', ... in the following way. Take any point O (called the *vertex of projection*) and any plane π' (called the *plane of projection*). Then A', B', C', ... and l', m', n', ... are the points and lines in which the plane of projection meets the lines and planes joining the vertex of projection to A, B, C, ... and l, m, n, ... Each of the figures ϕ and ϕ' is called the *projection* of the other; and they are said to be *in projection.*

Also each of the points A and A' is said to be the projection of the other; so for the points B and B', C and C', &c., and for the lines l and l', m and m', n and n', &c. The line in which the planes of the figures ϕ and ϕ' meet may be called the *axis of projection.*

When the vertex of projection is at infinity we get what is called *parallel projection*; in this case all the lines AA', BB', CC', ... are parallel. A particular case of parallel projection is *orthogonal projection.*

The lines AA', BB', CC', ... are called the *rays* of the projection; and projection is sometimes called *radial projection* to distinguish it from orthogonal projection.

Figures in projection are also said to be in perspective in different planes; and then the vertex of projection is called the *centre of perspective,* and the axis of projection is called the *axis of perspective,* and each figure is called the *perspective* or *picture* of the other. Note that figures may also be in perspective in the same plane. (See XXXI.)

Some writers use the term *conical projection* or *central projection* or *central perspective* for radial projection.

All this is illustrated in the model at end of the book. If the reader places the plane ρ parallel to the plane π', he will find that the line OA cuts the plane π' at A', the projection of A, and so on; also that the plane joining O to the line l cuts π' in the line l', the projection of l, and so on. The figure in the model is the triangle ABC which is projected into the triangle $A'B'C'$.

Note that the arrow-head placed near X', Y', Z' denotes that they are at infinity.

2. *The projection of the join of two points A, B is the join of the projections A', B' of the points A, B.*

The projection of the meet of the two lines l, m is the meet of the projections l', m' of the lines l, m.

The projection of any point on the axis of projection is the point itself.

Every line and its projection meet on the axis of projection.

The proofs of these four theorems are obvious. They should be verified on the model.

The projection of a tangent to a curve γ at a point A is the tangent at A' (the projection of A) to the curve γ' (the projection of γ).

For when the chord AB of γ becomes the tangent at A to γ by B moving up to A, the chord $A'B'$ of γ' becomes the tangent at A' to γ' by B' moving up to A'.

The projection of a meet (i.e. a common point) of two curves is a meet of the projections of the curves.

The projection of a common tangent to two curves is a common tangent to the projections of the curves.

The proofs of these theorems are obvious.

3. The plane through the vertex of projection parallel to the plane of one of two figures in projection meets the plane of the other figure in a line called the *vanishing line* of this plane. For instance, in the model, the line ZX is the vanishing line of the plane π if the plane ρ be placed parallel to the plane of projection π'.

Each vanishing line is parallel to the axis of projection.

For the axis of projection and the vanishing line in the plane π are the meets of π with π' and with the plane through O parallel to π'.

The vanishing line in one plane is the projection of the line at infinity in the other plane.

For the plane joining O to the vanishing line is parallel to the other plane.

To project a given line to infinity.

With any vertex of projection, project on to any plane parallel to the plane containing the given line and the vertex of projection. Then the projection of the given line will be the intersection of these two parallel planes and will therefore be entirely at infinity.

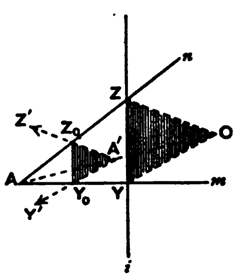
To project a given point to infinity, project any line through it to infinity.

To project any two lines into parallel lines, project their intersection to infinity.

4. The *vanishing point* of a line is the point in which the line meets the vanishing line of its own plane.

The angle between the projections of any two lines m and n is equal to the angle which the vanishing points of m and n subtend at the vertex of projection.

Let m and n meet in A, and let m meet the vanishing line i in Y and let n meet i in Z. We have to show that the projection of the angle YAZ is equal to YOZ, O being the vertex of projection. Now the plane of projection π' is parallel to the plane YOZ. Also $A'Z'$ is the meet of the plane AOZ and π'. Hence $A'Z'$ and OZ (being the meets of the plane AOZ with the two parallel planes π' and YOZ) are parallel. Similarly $A'Y'$ and OY are parallel. Hence $\angle Y'A'Z' = \angle YOZ$.



The reader should follow this proof both in the figure and in the model.

Notice that *all angles whose bounding lines have the same vanishing points are projected into equal angles.*

5. *To project any two given angles into angles of given magnitudes and at the same time any given line to infinity.*

Let the given angles ABC, DEF meet the line which is to be projected to infinity in AC, DF. Then, since A, C are the vanishing points of the lines BA, BC, the angle $A'B'C'$ is equal to AOC; so $\angle D'E'F' = \angle DOF$. Hence to construct O draw on AC a segment of a circle containing an angle equal to the given angle $A'B'C'$, and on DF and on the same side of it as before describe a segment of a circle containing an angle equal to the given angle $D'E'F'$. Let these segments meet in O. Rotate O about $ACDF$ out of the plane of the paper. Then if we project with vertex O on to a plane parallel to the plane $OACDF$, the problem is solved. For the line $ADCF$ will go to infinity. Also ABC will be projected into an angle equal to AOC, i. e. into an angle of the required size. So for DEF.

The segments may meet in two real points or in one or in none. Hence there may be two real solutions of the problem or one or none.

Ex. *In the exceptional case when the vanishing line is parallel to one of the lines of one of the angles, give a construction for the vertex of projection.*

Let A be at infinity. Through C draw a line making with CF the supplement of $A'B'C'$. This will meet the segment on DF in O.

6. *Given a line i and a triangle ABC, to project i to infinity and each of the angles A, B, C into an angle of given size.*

Suppose we have to project A, B, C into angles equal to α, β, γ, where of course $\alpha + \beta + \gamma = 180°$. Let i cut BC, CA,

AB in X, Y, Z. Of the points X, Y, Z let Y be the point which lies between the other two. On YZ describe a segment of a circle containing an angle equal to a. On XY and on the same side of i describe a segment of a circle containing an angle equal to γ. These two segments meet in Y; hence they meet again in another point, O, say. For if the supplements of the segments meet in O, then $ZOY + YOX = 180° - a + 180° - \gamma = 180° + \beta > 180°$, which is impossible. Now rotate O about i out of the plane of the paper. With O as vertex of projection, project on to any plane parallel to OXY; and let $A'B'C'$ be the projection of ABC.

We have to prove that $A' = a$, $B' = \beta$, $C' = \gamma$. Through Z draw a parallel to OX meeting OY in P. Then $ZOP = a$, $OPZ = \gamma$, and $PZO = \beta$. Also $A'B'$ is parallel to OZ, $B'C'$ is parallel to OX and therefore to ZP, and $C'A'$ is parallel to OY. Hence the sides of the triangles $A'B'C'$ and OZP are parallel. Hence the angles are equal; i.e. $A' = a$, $B' = \beta$, $C' = \gamma$.

The reader should perform this construction on a copy of the model.

7. *To project any triangle into a triangle with given angles and sides and any line to infinity.*

Project as above the given triangle ABC into $A'B'C'$ in which $\angle A' = \angle a'$, $\angle B' = \angle b'$, $\angle C' = \angle c'$, $a'b'c'$ being the triangle into which ABC is to be projected. On OA' take a point P such that $OP : OA' :: b'c' : B'C'$. Through P draw a plane parallel to $A'B'C'$ cutting OB' in Q and OC' in R. Then by similar triangles $OP : OA' :: QR : B'C'$; hence $QR = b'c'$. So $RP = c'a'$, $PQ = a'b'$. Hence PQR is superposable to $a'b'c'$ and in projection with ABC.

Hence we can *project any triangle into an equilateral triangle of any size and any line to infinity.*

Ex. 1. *Project any four given points into the angular points of a square of given size.*

Let $ABCD$ (II. 12) be the given points. Project UV to

infinity and the angles VAU, LWM into right angles. Then in the projected figure AB and CD are parallel, and also AD and BC. Also BAD is a right angle and also AWD. Hence the figure is a square. We can change its size as before. The construction is always real since the semicircles on LM and UV must meet since LM and UV overlap.

Ex. 2. *Project any two homological triangles (see § 11) simultaneously into equilateral triangles. Is the construction always real?*

Ex. 3. *Project any three angles into right angles.*

Let the legs of the angles A and B meet in L and M, and let LM cut the legs of C in DE; then on LM and DE describe semicircles.

Ex. 4. *Project any five points A, B, C, G, H into points A', B', C', G', H' such that G' shall be the centroid and H' the orthocentre of the triangle $A'B'C'$.*

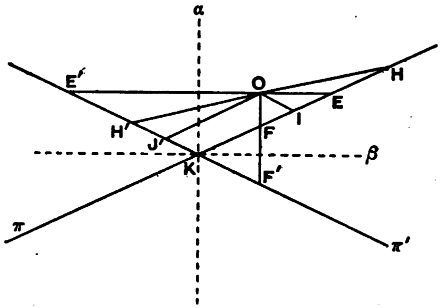
Let AG and BG meet BC and CA at D and E and take I and J such that (BC, DI) and (CA, EJ) are harmonic. Take IJ as vanishing line. Let AH and BH cut IJ at P and Q. The vertex of projection is the intersection of circles on PI and QJ as diameters.

Ex. 5. *Project any four points A, B, C, G into points A', B', C', G' such that $A'B'C'$ shall be an equilateral triangle of given size of which G' is the centroid.*

***8.** *In projecting from one plane to another, there are in each plane two points such that every angle at either of them is projected into an equal angle.*

Let the given planes be π and π'. Draw the planes a and β bisecting the angles between the planes π and π'. Through the vertex of projection O draw a line perpendicular to a cutting the planes π and π' in E, E', and a line through O perpendicular to β cutting the planes π and π' in F, F'. Then every angle at E will be projected into an equal angle at E', and every angle at F will be projected into an equal angle at F'.

The figure is a section of the solid figure by a plane through O perpendicular to the planes π and π'. Let this plane meet the axis of projection in K, and let the legs of any angle at E in π meet the axis of projection in L, M. Then the angle LEM projects into the angle $LE'M$.



But $EK = E'K$ by construction and $\angle EKL = \angle E'KL = 90°$. Hence the figure $EKLM$ is superposable to the figure $E'KLM$. Hence the angle LEM is equal to the angle $LE'M$, i.e. any angle at E is projected into an equal angle at E'. So any angle at F is projected into an equal angle at F'.

These points are called *equiangular points.*

There is (besides the axis of projection) in each plane a line segments on which are projected into equal segments.

For draw OI parallel to $E'F'$ and make $IH = KI$. Join HO cutting $E'F'$ at H'. Draw HP parallel to the axis of projection. Let PO cut the parallel through H' to the axis of projection at P'. Then, by parallels, $HO = OH'$ and OHP and $OH'P'$ are right angles. Hence $HP = H'P'$. So if we take Q on HP, we get $HQ = H'Q'$. Hence $PQ = P'Q'$; i.e. every segment on PH projects into an equal segment on $P'H'$.

These lines are called *equisegmental lines.*

Notice that the *equisegmental lines are the reflexions of the axis of projection in the vanishing lines;* for $KI = IH$ and similarly for H'.

9. *The projection of a harmonic range is a harmonic range.*

For if $A'B'C'D'$ be the projection of the harmonic range

$ABCD$, then O and the lines AB, $A'B'$ lie in one plane. Hence by II. 7.

The projection of a harmonic pencil is a harmonic pencil.

Draw any line cutting the rays of the harmonic pencil $U(ABCD)$ in a, b, c, d. Let $U'(A'B'C'D')$ be the projection of the pencil $U(ABCD)$, and a', b', c', d' the projections of a, b, c, d. Then a being on UA, a' is on $U'A'$, and so on; hence $U'(A'B'C'D')$ is harmonic, if $(a'b'c'd')$ is harmonic. And $(a'b'c'd')$ is harmonic, since $(abcd)$ is harmonic.

10. *To prove by projection the harmonic property of a complete quadrilateral.*

In the figure of II. 10, project one side of the harmonic triangle, say, βa to infinity. Then in the new figure AB' and $A'B$ meet at infinity, i.e. are parallel. So AB and $A'B'$ are parallel. Hence $ABA'B'$ is a parallelogram. Hence γ bisects BB'. Also a is at infinity. Hence $(BB', \gamma a)$ is harmonic in the new figure and therefore in the original figure. So $(AA', \gamma\beta)$ is harmonic. Hence $A'(BB', \gamma a)$ is harmonic. Hence its section $(CC', \beta a)$ is harmonic.

To prove by projection the harmonic property of a complete quadrangle.

In the figure of II. 12, project one side of the harmonic triangle, say, VU to infinity. Then $ABCD$ becomes a parallelogram. Hence WU bisects AD and WV is parallel to AD. Hence $W(AD, VU)$ is harmonic. So by projecting the other sides of the harmonic triangle to infinity, we prove that the pencils at U and V are harmonic.

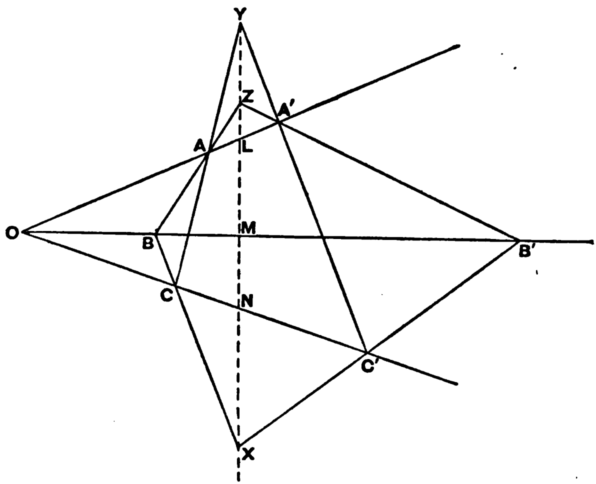
Ex. *We know* (III. 15, Ex. 3) *that the middle points of the three diagonals of a quadrilateral lie on a line. What does this proposition become if we project this line to infinity?*

In the figure of II. 10, let X, Y, Z bisect AA', BB', CC'; and let the points at infinity on AA', BB', CC' be X', Y', Z'. Then X', Y', Z' lie on the line at infinity i. Hence, on projecting, X', Y', Z' lie on a line not at infinity. Also in the original figure (AA', XX') is harmonic. Hence in the new figure (AA', XX') is harmonic. Also X is now at infinity. Hence X' bisects AA'. Hence we get the same

proposition, viz. that the three middle points X', Y', Z' are collinear.

Homological Triangles.

11. *Two triangles ABC, $A'B'C'$* are said to be *homological* (or in perspective) when AA', BB', CC' meet in a point (called the *centre of homology* or centre of perspective) and also $(BC; B'C')$, $(CA; C'A')$, $(AB; A'B')$ lie on a line (called the *axis of homology* or the axis of perspective).



If two triangles in the same plane be copolar, they are coaxal; and if coaxal, they are copolar.

(i) Let the two triangles ABC, $A'B'C'$ be copolar, i.e. let AA', BB', CC' meet in the point O; then they are coaxal, i.e. $(BC; B'C')$, $(CA; C'A')$, $(AB; A'B')$ lie on a line.

Call these three points X, Y, Z. Then we have to show that YZ passes through X. Project YZ to infinity. Then in the new figure AA', BB', CC' meet in a point O; also AB is parallel to $A'B'$ and AC to $A'C'$. Hence

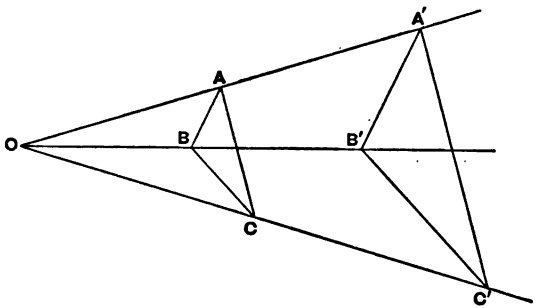
$$OB : OB' :: OA : OA' :: OC : OC'.$$

And since $OB : OB' :: OC : OC'$, BC is parallel to $B'C'$, i.e. X is at infinity, i.e. X lies on YZ, i.e. XYZ are collinear.

Hence in the original figure XYZ are collinear, i.e. the triangles are coaxal.

(ii) Let the triangles be coaxal, i.e. let $(BC; B'C')$, $(CA; C'A')$, $(AB; A'B')$ be collinear; then they are copolar, i.e. AA', BB', CC' meet in a point.

Project XYZ to infinity. Then in the new figure BC is parallel to $B'C'$, CA to $C'A'$, and AB to $A'B'$. Let AA' and BB' meet in O. Then $OB : OB' :: AB : A'B' :: BC : B'C'$; and $\angle OBC = \angle OB'C'$. Hence the triangles OBC and $OB'C'$



are similar. Hence $\angle BOC = \angle B'OC'$. Hence CC' passes through O. Hence AA', BB', CC' meet in a point. Hence AA', BB', CC' meet in a point in the original figure.

***12.** If the triangles are not in one plane, the proofs are simpler.

If two triangles be copolar, they are coaxal.

(Use the same figure as before, but remember that now the triangles are in different planes; or use the model.) Since AB, $A'B'$ lie in the plane $OAA'BB'$, hence AB, $A'B'$ meet in a point on the meet of the planes ABC, $A'B'C'$. Similarly $(CA; C'A')$, $(BC; B'C')$ lie on this line, i.e. the triangles are coaxal.

If two triangles be coaxal, they are copolar.

The three planes $BCXB'C'$, $CAYC'A'$, $ABZA'B'$ meet in a point; hence their meets AA', BB', CC' pass through this point, i.e. the triangles are copolar.

This gives us another proof of the theorem of § 11. For since the theorem is true however near the planes are to one another, it is true when they coincide.

Ex. 1. *If three triangles ABC, $A'B'C'$, $A''B''C''$, which are homologous in pairs, be such that BC, $B'C'$, $B''C''$ are concurrent and CA, $C'A'$, $C''A''$ and AB, $A'B'$, $A''B''$; then the three centres of homology of the triangles taken in pairs are collinear.*

For the triangles $AA'A''$, $BB'B''$ are copolar and therefore coaxal.

Ex. 2. *If three triangles ABC, $A'B'C'$, $A''B''C''$ be such that $AA'A''$, $BB'B''$, $CC'C''$ are concurrent lines; then the axes of homology of the triangles taken in pairs are concurrent.*

For the triangles whose sides are AB, $A'B'$, $A''B''$ and AC, $A'C'$, $A''C''$ are coaxal and therefore copolar.

Ex. 3. *The triangles ABC, $A'B'C'$ are coaxal; if $(BC; B'C')$ be X, $(CA; C'A')$ be Y, $(AB; A'B')$ be Z, $(BC'; B'C)$ be X', $(CA'; C'A)$ be Y', $(AB', A'B)$ be Z'; then $XY'Z'$, $X'YZ'$, $X'Y'Z$ are lines.*

IV

1. If the points A', B', C' lie on the lines BC, CA, AB, and if AA', BB', CC' meet in a point, show that the intersections of BC, $B'C'$, of CA, $C'A'$ and of AB, $A'B'$ lie on a line which bisects the lines drawn from A, B, C to BC, CA, AB parallel to $B'C'$, $C'A'$, $A'B'$.

2. Project the four-sided figure $ABCD$ into an equilateral figure with given angles.

3. Through the intersection of the diagonals of a quadrilateral lines are drawn parallel to each of the four sides to meet the opposite sides. Show that the four intersections are collinear.

CHAPTER V

HARMONIC PROPERTIES OF A CONIC

1. WE define a *conic section* or briefly a *conic* as the projection of a circle, or in other words, as the plane section of a cone on a circular base. The plane of projection may be called the plane of section.

From the definition of a conic it immediately follows that—

Every line meets a conic in two points, real, coincident, or imaginary.

From every point can be drawn to a conic two tangents, real, coincident, or imaginary.

For these properties are true for a circle, and therefore for a conic by projection.

2. There are three kinds of conics according as the vanishing line meets the circle, touches the circle, or does not meet the circle, or more properly according as the vanishing line meets the circle in real, coincident, or imaginary points.

If the vanishing line meet the circle in two points P and Q, then, V being the vertex of projection, the plane of section is parallel to the plane VPQ, and therefore cuts the cone on both sides of V. Hence we get a conic consisting of two detached portions, extending to infinity in opposite directions, called a *hyperbola.*

If the vanishing line touch the circle, and TT' be the tangent, then the plane of section, being parallel to the plane VTT' which touches the cone, cuts the cone on one side only of V. Hence we get a conic consisting of one portion extending to infinity, called a *parabola.*

If the vanishing line does not meet the circle, the plane of section is parallel to a plane through V which does not

meet the cone except at the vertex, and therefore cuts the cone in a single closed oval curve, called an *ellipse.*

Since the line at infinity is the projection of the vanishing line, it follows that the line at infinity meets a hyperbola in two points, touches a parabola, and does not meet an ellipse; in other words, *the line at infinity meets a hyperbola in two real points, a parabola in two coincident points, and an ellipse in two imaginary points,* or, again, a hyperbola has two real points at infinity, a parabola two coincident points, and an ellipse two imaginary points.

3. *A pair of straight lines is a conic.*

For let the cutting plane be taken through the vertex, so as to cut the cone in two lines. Then these lines are a section of the cone, i.e. a conic.

But properties of a pair of lines cannot be directly obtained by projection from a circle. For let the cutting plane meet the circle in the points P and Q. Then the projection of every point on the circle except P and Q is at the vertex, whilst the projection of P is any point on the line VP and the projection of Q is any point on the line VQ. Now if we take any point R' on one of the lines VP and VQ, its projection is P or Q unless R' is at the vertex and then its projection is some point on the rest of the circle.

To get over this difficulty we take a section of the cone parallel to the section through the vertex. Then however near the vertex this plane is, the theorem is true for the hyperbolic section; hence the theorem is true in the limit when the section passes through the vertex and the hyperbola becomes a pair of lines.

4. *A pair of points is a conic.*

This follows by Reciprocation. (See VIII.) For the reciprocal of two lines is two points and the reciprocal of a conic is a conic. Hence two points is a conic.

Clearly, however, we cannot obtain two points by the section of a circular cone.

The reader should notice that a pair of lines and a pair of

points are only conics in a restricted sense; for we cannot draw a pair of tangents from a point to a pair of lines and a line cannot cut a pair of points in two points. In fact a pair of lines is only a conic when considered as a locus and a pair of points is only a conic when considered as an envelope.

5. As in the case of the circle we define *the polar of a point for a conic* as the locus of the fourth harmonics of the point for the conic.

The polar of a point for a conic is a line.

Through the given point U draw a chord PP' of the conic and on this chord take the point R, such that (PP', UR) is harmonic. We have to show that the locus of R is a line. Now by hypothesis the conic is the projection of a circle. Suppose the range (PP', UR) is the projection of (pp', ur) in the figure of the circle. Then, since (PP', UR) is harmonic, so is (pp', ur). Hence r is on the locus of the fourth harmonics of u for the circle; hence the locus of r is a line. Hence by projection the locus of R is a line.

As in the case of the circle, if the line u is the polar of U for a conic, then U is defined to be the *pole* of u for the conic; and U and u are said to be *pole and polar* for the conic.

We have proved above implicitly that *The projection of a pole and polar for a circle is a pole and polar for the conic which is the projection of the circle.*

The following theorems now follow at once by projection.

If P be outside the conic, the polar of P is the chord of contact of tangents from P.

If P be on the conic, the polar of P is the tangent at P, and the pole of a tangent is the point of contact.

Note that a point is said to be inside or outside a conic according as the tangents from the point are imaginary or real, i.e. according as the polar of the point meets the curve in imaginary or real points. When the point is on the conic, its polar, viz. the tangent, meets the curve in coincident

points and the tangents from the point coincide with the tangent at the point.

Ex. 1. *A chord PQ of a conic is drawn through the fixed point U, and u is the polar of U; show that $(P, u)^{-1} + (Q, u)^{-1}$ is constant.*

viz. $= 2 . (U, u)^{-1}$ by similar triangles.

Ex. 2. *From any point on the line u, tangents p and q are drawn to a conic, and U is the pole of u, and A is any point; show that*

$$\frac{(A, p)}{(U, p)} + \frac{(A, q)}{(U, q)} = 2 . \frac{(A, u)}{(U, u)}.$$

Take U on the range UA as origin.

6. Since a pole and polar project into a pole and polar, the whole theory of conjugate points and conjugate lines for a conic follows at once by projection from the theory of conjugate points and conjugate lines for a circle. Hence all the theorems enunciated in III. 10–12 for a circle follow for a conic by projection.

Ex. 1. *If a series of conics be drawn touching two given lines at given points, the polar of every point on the chord of contact is the same for all.*

Let the conics touch TL and TM at L and M. The polar of P on LM passes through T the pole of LM and passes through the fourth harmonic of P for LM.

Ex. 2. *The pole of any line through T is the same for all.*

Ex. 3. *TP, TQ touch a conic at P and Q, and on PQ is taken the point U such that TU bisects the angle PTQ, and through U is drawn any chord RUR' of the conic; show that TU also bisects the angle RTR'.*

Draw TU' perpendicular to TU; then TU' is the polar of U. Hence (ZU, RR') is harmonic, Z being on TU'.

Ex. 4. *Through the point U is drawn the chord PQ of a conic and UY is drawn perpendicular to the polar of U; show that UY bisects the angle PYQ or its supplement.*

Ex. 5. *The polar of any point taken on either of two lines which are conjugate for a conic meets the lines and the conic in pairs of harmonic points.*

For if P be the point, its polar meets the other line in the pole of the line on which P is.

Ex. 6. *A, B, C are three points on a conic and CT is the*

tangent at C; if $C\,(TD, AB)$ be harmonic, show that CD passes through the pole of AB.

Let AB, CT meet at R. Then the polar of R passes through C and through the fourth harmonic of R for AB; and hence is CD. Hence CD passes through the pole of AB since R is on AB.

Ex. 7. *TP, TQ touch a conic at P, Q; the tangent at R meets PQ in N, PT in L, QT in M; show that (LM, RN) is harmonic.*

Ex. 8. *Through U, the mid-point of a chord AB of a conic is drawn any chord PQ. The tangents at P and Q cut AB in L and M. Prove that $AL = BM$.*

If R be the pole of PQ, then $R\Omega$ is the polar of U, Ω being the point at infinity upon AB. Hence $UL = UM$.

Ex. 9. *The tangents TP, TP' to a conic are cut by the tangent at Q (which is parallel to the chord of contact PP') in L, L'; show that $LQ = QL'$.*

7. The theory of *self-conjugate triangles for a conic* follows at once by projection from a circle, since the theory involves only the theory of poles and polars.

Of the three vertices of a self-conjugate triangle two are outside and one inside the conic.

Let UVW be the vertices of the given triangle. Then if U is outside, VW, being the polar of U, cuts the conic. Also V, W form a harmonic pair with the meets of VW and the conic; hence V or W is outside the conic.

If U is inside, VW does not cut the conic, and hence V and W are both outside the conic.

Of the three sides of a self-conjugate triangle, two cut the conic and one does not.

For let U be inside and V, W outside the conic. Then VU and WU cut the conic. But VW does not; for its polar U is inside the conic.

8. *The harmonic points of a quadrangle inscribed in a conic form a triangle which is self-conjugate for the conic.*

The harmonic lines of a quadrilateral circumscribed to a conic form a triangle which is self-conjugate for the conic.

If a quadrilateral be circumscribed to a conic, the harmonic

triangle of this quadrilateral coincides with the harmonic triangle of the inscribed quadrangle formed by the points of contact.

For these propositions are true for the circle, and they follow for the conic by projection. So also—*the quadrangle construction for the polar of a point applies to a conic.*

Through a given point P draw a pair of tangents to a conic.

By the quadrangle construction obtain the polar of P for the conic, and join P to the points where this polar cuts the conic. The joining lines are the tangents from P to the conic.

Ex. 1. *A, B, C, D are four points on a conic; AB, CD meet in E, and AC, BD meet in H, and the tangents at A and D meet in G; show that E, G, H are collinear.*

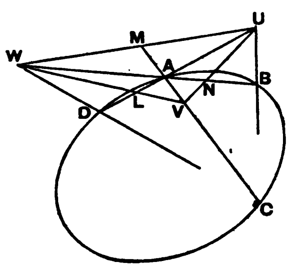
On the polar of the intersection of AD, BC.

Ex. 2. *A system of conics touch AB and AC at B and C. D is a fixed point and BD, CD meet one of the conics in P, Q. Show that PQ meets BC in a fixed point.*

Ex. 3. *The lines AB, BC, CD, DA touch a conic at a, b, c, d, and AB and CD are parallel. If ac, bd meet at E, and AD, BC meet at F, show that FE bisects AB and CD.*

For AC, BD also meet at E. Hence if AB and CD meet at Ω, then FE and $F\Omega$ are harmonic with FA and FB.

9. *If one point on a conic be given and also a triangle self-conjugate for the conic, then three other points are known.*



Let A be the given point and UVW the given self-conjugate triangle. Let UA cut WV in L. Then the other point D in which UA cuts the conic is known since $(UALD)$ is harmonic. Similarly the points C and B where VA and WA cut the conic are known.

The four points A, B, C, D form an inscribed quadrangle of which UVW is the harmonic triangle.

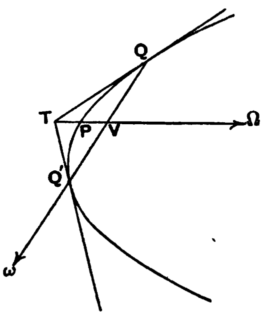
By construction $(UALD)$ is harmonic; hence $W(UAVD)$ is harmonic. Similarly, since $(MAVC)$ is harmonic, $W(UAVC)$ is harmonic. Hence WD and WC coincide,

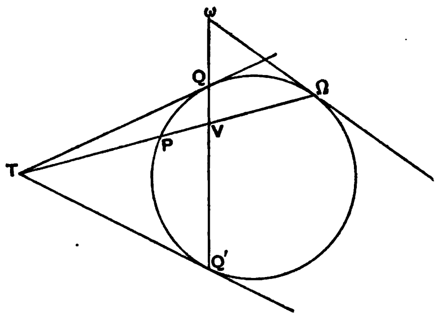
i. e. CD passes through W. Similarly CB passes through U and BD through V. Hence UVW is the harmonic triangle of the inscribed quadrangle $ABCD$.

In a similar manner, or better by reciprocating the above proposition (see Chapter VIII), we prove that *if one tangent of a conic is given and also a self-conjugate triangle, then three other tangents are known; and the four tangents form a circumscribed quadrilateral of which the given triangle is the harmonic triangle.*

Notice that we have incidentally proved that *if two sides of a triangle inscribed in a conic pass through two vertices of a triangle self-conjugate for a conic, then the third side passes through the third vertex* and the reciprocal theorem.

10. Properties peculiar to the parabola follow from the fact that the line at infinity touches the parabola.





The lines TQ, TQ' touch a parabola at Q, Q', and TV bisects QQ' in V and meets the curve in P; show that $TP = PV$.

Take the point at infinity ω on QQ'. Then since ω lies on the polar of T, hence the polar of ω passes through T. Since $(\omega V, QQ')$ is harmonic, hence the polar of ω passes through V. Hence TV is the polar of ω. Now suppose the line at infinity to touch the parabola in Ω. Then ω is on the polar of Ω, viz. the line at infinity; hence TV passes through Ω. Also P and Ω being points on the curve, therefore $(TV, P\Omega)$ is harmonic; hence $TP = PV$.

For clearness the figure is drawn of which the above figure is the projection. In this case, as in other cases, the theorem might have been proved directly by projection.

Ex. 1. *The line half-way between a point and its polar for a parabola touches the parabola.*

For the tangent at P passes through ω.

Ex. 2. *The lines joining the middle points of the sides of a triangle which is self-conjugate for a parabola touch the parabola.*

Ex. 3. *The nine-point circle of a triangle which is self-conjugate for a parabola passes through the focus.*

Ex. 4. *Through the vertices of a triangle circumscribing a parabola are drawn lines parallel to the opposite sides; show that these lines form a triangle self-conjugate for the parabola.*

Being the harmonic triangle of the circumscribing quadrilateral formed by the sides of the triangle and the line at infinity.

Ex. 5. *No two tangents of a parabola can be parallel.*

For if possible let them meet at ω on the line at infinity; then three tangents are drawn from ω to the conic, viz. the two tangents and the line at infinity.

11. We define the pole of the line at infinity for a conic as the *centre* of the conic. Hence *the centre of a parabola is at infinity.* For since the line at infinity touches the parabola, the centre is the point of contact and therefore is on the line at infinity, i.e. is at infinity. The centre of a hyperbola is outside the curve since the polar of the centre cuts the hyperbola in real points; and the centre of an ellipse is inside the curve since the polar of the centre cuts the ellipse in imaginary points. The hyperbola and ellipse are called *central conics.*

The centre of a central conic bisects every chord through it.

Let the chord PP' pass through the centre C of a conic; then $PC = CP'$. For let PP' meet the line at infinity in ω. Then since ω is on the polar of C, hence $(C\omega, PP')$ is harmonic. Hence $PC = CP'$.

A conic is its own reflexion in its centre.

For if we join any point P on the conic to the centre C

and produce PC backwards to P', so that $CP' = PC$; then P' is another point on the conic.

Ex. 1. *All conics circumscribing a parallelogram have their centres at the centre of the parallelogram.*

For by the quadrangle construction for a polar, the polar of the intersection of diagonals is the line at infinity.

Ex. 2. *QQ' is the chord of contact of tangents from T to a conic, and CT cuts QQ' in V and the conic in P; show that*

$$CV \,.\, CT = CP^2.$$

For (PP', TV) is harmonic.

Ex. 3. *Given the centre O of a conic and a self-conjugate triangle ABC, construct six points on the conic.*

12. *The locus of the middle points of parallel chords of a conic is a line* (called a *diameter*).

Let QQ' be one of the parallel chords bisected in V. The system of chords parallel to QQ' passes through a point ω at infinity. Also since $(\omega V, QQ')$ is harmonic, V is on the polar of ω. Hence the locus required is the polar of ω.

All diameters of a central conic pass through the centre.

All diameters of a parabola are parallel.

For since a diameter is the polar of a point on the line at infinity, it passes through the pole of the line at infinity. Hence in a central conic it passes through the centre, and in a parabola it passes through a fixed point at infinity, viz. the point of contact of the line at infinity.

Ex. 1. *The tangents at the ends of a diameter are parallel to the chords which the diameter bisects.*

Being the tangents from ω.

Ex. 2. *A diameter contains the poles of all the chords it bisects.*

Viz. the poles of lines through ω.

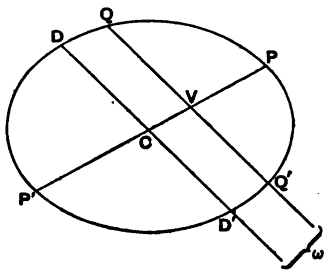
Ex. 3. *If the tangents at the ends of a chord are parallel, the chord is a diameter.*

Ex. 4. *Two chords of a conic which bisect one another are diameters.*

13. Conjugate lines at the centre of a conic are called *conjugate diameters.*

Each of two conjugate diameters bisects chords parallel to the other.

Let PCP' and DCD' be conjugate diameters. Then by definition the pole of CP is on CD. But CP passes through the centre; hence the pole of CP is at infinity. Hence the pole of CP is the point ω at infinity on CD. Through ω, i.e. parallel to DD', draw the chord QQ' meeting CP in V. Then since PP' is the polar of ω, hence $(QQ', V\omega)$ is harmonic, i.e. $QV = VQ'$. Hence PP' bisects every chord parallel to DD'. So DD' bisects every chord parallel to PP'.



A pair of conjugate diameters form with the line at infinity a self-conjugate triangle.

For if CP and CD are the conjugate diameters, the pole of CP lies on CD by hypothesis; it is also on the line at infinity since CP passes through C, the pole of the line at infinity. Hence the pole of CP is the intersection of CD with the line at infinity; so for CD.

In the case of an ellipse each of a pair of conjugate diameters meets the curve in real points; but in the case of a hyperbola, one meets the curve in real points and the other does not.

For we know that of the three sides of a self-conjugate triangle, two meet the curve and one does not. Also in the hyperbola the straight line at infinity cuts the curve and in the ellipse does not.

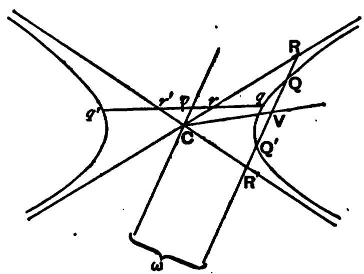
Ex. 1. *The line joining any point to the middle point of its chord of contact passes through the centre.*

Ex. 2. *The diagonals of a parallelogram circumscribing a conic are conjugate diameters; and the points of contact are the vertices of a parallelogram whose sides are parallel to the above diagonals.*

14. *If each diameter of a conic be perpendicular to its conjugate diameter, the conic is a circle.*

Take any two points P, Q on the conic. Bisect PQ in V and join CV. Then CV is the diameter bisecting chords parallel to PQ, i.e. CV and PQ are parallel to conjugate diameters. Hence CV and PQ are perpendicular. Also $PV = VQ$. Hence $CP = CQ$. Hence all radii of the conic are equal, i.e. the conic is a circle.

15. The *asymptotes* of a conic are the tangents from the centre. They are clearly the joins of the centre to the points at infinity on the conic. In the hyperbola they are real and distinct, in the parabola they coincide with the line at infinity, and in the ellipse they are imaginary. *The asymptotes are harmonic with every pair of conjugate diameters.* For the tangents from any point are harmonic with any pair of conjugate lines through the point.



Any line cuts off equal lengths between a hyperbola and its asymptotes.

Let a line cut the hyperbola in Q, Q' and its asymptotes in R, R'; then $RQ = Q'R'$.

On RR' take the point at infinity ω and bisect QQ' in V. Then since $(QQ', V\omega)$ is harmonic, the polar of ω passes through V. Since ω is at infinity, its polar passes through C. Hence CV is the polar of ω. Hence CV and $C\omega$ are conjugate lines. And CR, CR' are the tangents from C. Hence $C(RR', V\omega)$ is harmonic. Hence $(RR', V\omega)$ is harmonic. Hence $RV = VR'$. But $QV = VQ'$. Hence $RQ = Q'R'$. The proof applies whether we take QQ' to cut the same branch in two points or (as in the case of qq') to cut different branches of the hyperbola.

The intercept made by any tangent between the asymptotes is bisected at the point of contact.

For let Q and Q' coincide; then $RQ = QR'$.

The proofs of this article will be made clearer by drawing (as in § 10) the figure of the circle of which the given figure is the projection.

Ex. 1. *Two of the diagonals of a quadrilateral formed by two tangents of a hyperbola and the asymptotes are parallel to the chord joining the points of contact of the tangents.*

Consider the figure of the circle of which the given figure is the projection. The solution then depends on III. 18 (end).

· **Ex. 2.** *If a hyperbola be drawn through two opposite vertices of a parallelogram with its asymptotes parallel to the sides, show that the centre lies on the join of the other vertices.*

16. *A rectangular hyperbola* is defined to be a hyperbola whose asymptotes are perpendicular.

Conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

For they form a harmonic pencil with the asymptotes, which are perpendicular.

Ex. *The lines joining the ends of any diameter of a rectangular hyperbola to any point on the curve are equally inclined to the asymptotes.*

17. *A principal axis* of a conic is a diameter which bisects chords perpendicular to itself.

All conics have a pair of principal axes; but one of the principal axes of a parabola is at infinity.

Consider first the *hyperbola.* Then the asymptotes are real and distinct. Now the bisectors of the angles between the asymptotes are harmonic with the asymptotes and are therefore conjugate diameters. But the bisectors are also perpendicular. Hence they are a pair of conjugate diameters at right angles. Each of the bisectors is therefore a principal axis; for each bisects chords parallel to the other, i. e. perpendicular to itself.

Also since the axes are conjugate diameters, one meets the curve and the other does not. The former is called the *transverse axis* and the other the *conjugate axis.*

Consider next the *parabola.* We might say that here the

asymptotes are coincident with the line at infinity; and the bisectors of the angles between a pair of coincident lines are the line with which they coincide and a perpendicular to it. Hence the principal axes of a parabola are the line at infinity and another line called *the axis of the parabola.*

Or thus—All the diameters of a parabola are parallel. Draw chords perpendicular to a diameter, then the diameter bisecting these chords is perpendicular to them and is called *the axis of the parabola.* The other principal axis (like the diameter conjugate to any of the other parallel diameters) is the line at infinity.

Consider last *the ellipse.* Here the asymptotes are imaginary and this method fails. But it will be proved under Involution that there is always a pair of conjugate diameters of any conic at right angles. Hence the ellipse also has a pair of principal axes. (See XIX. 4.)

Since the axes are conjugate diameters, each of the axes meets the curve in real points.

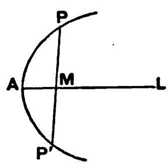
An axis cuts the conic at right angles.

For the tangent at the end of an axis is the limit of a bisected chord.

A central conic is symmetrical for each axis.

For the principal axis AL bisects chords perpendicular to itself.

Let PMP' be such a chord. Then P' is clearly the reflexion of P in AL, i.e. the conic is symmetrical for AL.



The same proof shows that

A parabola is symmetrical for its axis.

Ex. 1. *The tangent at P meets the axis CA in T and PN is the perpendicular on CA; show that $CN \cdot CT = CA^2$.*

For PN is the polar of T.

Ex. 2. *PQ, PR touch a conic at Q, R. PM is drawn perpendicular to either axis. Show that PM bisects the angle QMR.*

V

1. A is a fixed point, P is a point on the polar of A for a given conic. The tangents from P meet a fixed line at Q, R.

AR, PQ meet at X; and AQ, PR at Y. Show that XY is a fixed line.

2. A and B are two fixed points. A line through A cuts a fixed conic at C and D, BD cuts the polar of A at F, and BC cuts the polar at E. Show that DE and CF meet at a fixed point.

3. Through the fixed point A is drawn the variable chord PQ of a conic and the chords PU, QV pass through the fixed point B. Show that UV passes through a fixed point.

4. Through the point C are drawn the chords PP', QQ' and the tangents CA, CB of a conic. Show that a conic which touches the four lines PQ, $P'Q'$, $P'Q$, PQ' and passes through B, touches BC at B.

5. ABC is a triangle circumscribed to a conic, and the point P of contact of BC bisects BC. Show that the centre of the conic is on AP.

6. Through the point O are drawn the chords PP', QQ' of a conic and any line through O cuts the conic at L, L' and cuts PQ, $P'Q'$ at M, M'. Show that

$$\frac{1}{OL}+\frac{1}{OL'}=\frac{1}{OM}+\frac{1}{OM'}.$$

7. F and G are conjugate points on the chords PQ and PR of a conic. Show that FG and QR are conjugate lines.

8. Two conics, c_1 and c_2, touch at A and B. A line through A cuts c_1 at P and c_2 at Q, and a line through B cuts c_1 at P' and c_2 at Q'. Show that PP' and QQ' meet on AB.

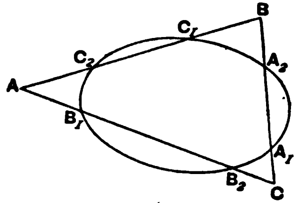
CHAPTER VI

CARNOT'S THEOREM

1. *The sides BC, CA, AB of a triangle cut a conic in the points A_1A_2, B_1B_2, C_1C_2, show that*

$$AC_1 . AC_2 . BA_1 . BA_2 . CB_1 . CB_2 = AB_1 . AB_2 . BC_1 . BC_2 . CA_1 . CA_2.$$

By definition a conic is the projection of a circle. Let the points $ABCA_1A_2 \ldots$ be the projections of $A'B'C'A_1'A_2' \ldots$ in the figure of the circle.

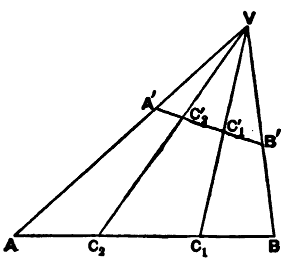


Now in the circle we have

$$A'C_1' . A'C_2' . B'A_1' . B'A_2' . C'B_1' . C'B_2' = A'B_1' . A'B_2' . B'C_1' . B'C_2' . C'A_1' . C'A_2'$$

for $A'C_1' . A'C_2' = A'B_1' . A'B_2'$, and so on.

Let V be the vertex of projection.



Then $$\frac{AC_2}{BC_2} = \frac{\Delta AVC_2}{\Delta BVC_2} = \frac{AV . C_2V . \sin AVC_2}{BV . C_2V . \sin BVC_2} = \frac{AV}{BV} . \frac{\sin AVC_2}{\sin BVC_2}$$

and so for each ratio.

Hence $$\frac{AC_1 . AC_2 \ldots}{AB_1 . AB_2 \ldots} = \frac{\sin AVC_1 . \sin AVC_2 \ldots}{\sin AVB_1 . \sin AVB_2 \ldots}$$

where each segment is replaced by the sine of the corresponding angle. Also the last expression

$$= \frac{\sin A'VC_1' . \sin A'VC_2' \ldots}{\sin A'VB_1' . \sin A'VB_2' \ldots}, \text{ and this equals } \frac{A'C_1' . A'C_2' \ldots}{A'B_1' . A'B_2' \ldots},$$

by exactly the same reasoning as before, and this has been proved equal to unity. Hence

$$AC_1 . AC_2 . BA_1 . BA_2 . CB_1 . CB_2$$
$$= AB_1 . AB_2 . BC_1 . BC_2 . CA_1 . CA_2 .$$

The reader should notice that this relation is homogeneous both in the points and in the lines of the figure. For if we replace AC_1 by $A \times C_1$ and so on, the points cross out; and if we replace AC_1 by the line c on which it lies and so on, the lines cross out.

Any relation connecting segments on lines is unaltered by projection if it is homogeneous both in the points and in the lines of the figure.

For take any segment AB and draw the perpendicular p from V on AB. Then $p . AB = VA . VB \sin AVB$, each being twice the area of the triangle AVB. Hence $AB = VA . VB \sin AVB \div p$. Now replace each segment such as AB by $VA . VB \sin AVB \div p$. Then since the given relation is homogeneous in the points, VA, VB etc. will cross out; and since it is homogeneous in the lines, the perpendiculars p etc. will cross out, leaving us with a relation containing only the sines $\sin AVB$ etc. Now, since $AVB = A'VB'$, this relation holds in the new figure; and leads to the same relation between $A'B'$ etc. as was given between AB etc.

Ex. 1. *The sides AB, BC, CD,... of a polygon meet a conic in $A_1 A_2$, $B_1 B_2$, $C_1 C_2$,...; show that*

$$AA_1 . AA_2 . BB_1 . BB_2 . CC_1 . CC_2 ...$$
$$= BA_1 . BA_2 . CB_1 . CB_2 . DC_1 . DC_2 ....$$

Ex. 2. *If a conic touch the sides of the triangle ABC in A_1, B_1, C_1; then AA_1, BB_1, CC_1 are concurrent.*

For $AB_1^2 . CA_1^2 . BC_1^2 = AC_1^2 . BA_1^2 . CB_1^2$; and we cannot have $AB_1 . CA_1 . BC_1 = +AC_1 . BA_1 . CB_1$, for then $A_1 B_1 C_1$ would cut the conic in three points.

Ex. 3. *If the vertex A in Carnot's theorem be on the conic, show that the ratio $AC_2 : AB_1$ must be replaced by* $\sin TAC : \sin TAB$, *AT being the tangent at A.*

For $B_1 C_2$ is ultimately the tangent at A.

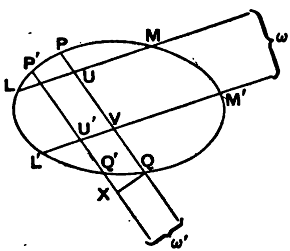
Ex. 4. *What does Carnot's theorem reduce to when A, B, and C are on the curve?*

Ex. 5. *If through fixed points A, B we draw the chords AB_1B_2, BA_2A_1 of a conic meeting in the variable point C, then the ratio $BA_1 . BA_2 . CB_1 . CB_2 \div AB_1 . AB_2 . CA_1 . CA_2$ is constant.*

Ex. 6. *Deduce the corresponding theorem when B is at infinity.*

Ex. 7. *A conic cuts the sides BC, CA, AB of a triangle in P_1P_2, Q_1Q_2, R_1R_2; BQ_2 and CR_2 meet in X, AP_1 and CR_1 in Y, and AP_2 and BQ_1 in Z; show that AX, BY, CZ are concurrent.*

2. Newton's theorem—*If two chords of a conic UPQ, ULM be drawn in given directions through a variable point U, show that the ratio of $UP . UQ$ to $UL . UM$ is independent of the position of U.*



Let $U'P'Q'$, $U'L'M'$ be another position of the chords UPQ, ULM. Then PQ is parallel to $P'Q'$ and LM to $L'M'$. Let PQ, $P'Q'$ meet at infinity in ω', and LM, $L'M'$ at infinity in ω. Apply Carnot's theorem to the triangle $\omega'U'V'$. Then

$$\omega'Q' . \omega'P' . U'L' . U'M' . VQ . VP = \omega'Q . \omega'P . VL' . VM' . U'Q' . U'P'.$$

From Q drop the perpendicular QX on $Q'\omega'$.

Then $\omega'Q'/\omega'Q = (\omega'X + XQ')/\omega'X = 1 + XQ'/\omega'X = 1$.

So $\omega'P' = \omega'P$. Hence

$$U'L' . U'M' . VQ . VP = VL' . VM' . U'Q' . U'P'$$

i.e. $$U'P' . U'Q' \div U'L' . U'M' = VP . VQ \div VL' . VM'$$

In exactly the same way the triangle ωUV gives us

$$VP . VQ \div VL' . VM' = UP . UQ \div UL . UM.$$

Hence $UP . UQ \div UL . UM = U'P' . U'Q' \div U'L' . U'M'$,

i.e. $UP . UQ \div UL . UM$ is independent of the position of U.

If TN and TR are the tangents and XCX' and YCY' the diameters parallel to UPQ and ULM, then

$$UP . UQ \div UL . UM = TN^2 \div TR^2 = CX^2 \div CY^2.$$

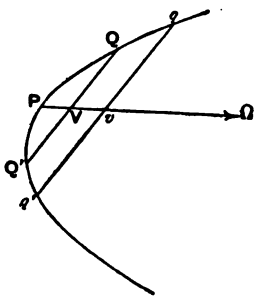
For, taking U successively at T and C, we have
$UP.UQ \div UL.UM = TN.TN \div TR.TR$
$= CX.CX' \div CY.CY' = -CX^2 \div -CY^2$.

Ex. ***In any conic, any of the pairs of chords joining the intersections of the conic and any circle are equally inclined to each axis of the conic.***

For $UP.UQ = UL.UM$; hence $CX = CY$.

3. ***In a parabola*** $QV^2 = 4.SP.PV$.

Besides QVQ' draw a second double ordinate qvq' of the diameter PV. Now PV meets the parabola again at Ω, a point at infinity. Also by Newton's theorem we have



$$\frac{VQ.VQ'}{VP.V\Omega} = \frac{vq.vq'}{vP.v\Omega}.$$

But $V\Omega = v\Omega$. Hence $VQ.VQ' \div VP = vq.vq' \div vP$, i.e. $QV^2 \div PV$ is constant. To obtain the value of this constant, take qq' through the focus S. Then by Geometrical Conics $qq' = 4.SP$ and $Pv = SP$. Hence $QV^2 \div PV = 4.SP$.

Note that the theorem also follows directly from Carnot's theorem by using the triangle contained by QV, Vv, vq.

If we take PV to be the axis of the parabola, the relation becomes $QV^2 = 4.SA.AV$, A being the vertex of the parabola, and QV being perpendicular to the axis; or, as it is more commonly written, $PN^2 = 4.AS.AN$.

This is called the equation of the parabola.

4. ***In an ellipse*** $QV^2 : PV.VP' :: CD^2 : CP^2$.

In the figure of V. 13, we have by Newton's theorem,

$$VQ.VQ' : VP.VP' :: CD.CD' : CP.CP',$$
$$\text{i.e. } QV^2 : PV.VP' :: CD^2 : CP^2.$$

If we take CV to be an axis of the ellipse the relation becomes $QV^2 : AV.VA' :: CB^2 : CA^2$, QV being now perpendicular to the axis ACA'; or as it is more commonly

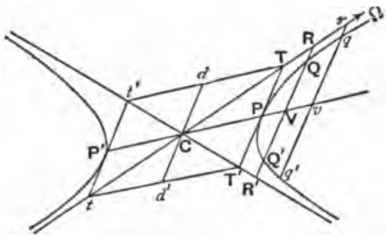
written $PN^2 : AN.NA' :: CB^2 : CA^2$ which is equivalent to

$$\frac{CN^2}{CA^2} + \frac{PN^2}{CB^2} = 1.$$

This is called the equation of the ellipse.

5. *In a hyperbola* $QV^2 : PV.VP' :: CD^2 : CP^2$.

Besides QVQ' draw a second double ordinate qvq' of the diameter PCP'. Then by Newton's theorem $VQ.VQ' : VP.VP' :: vq.vq' : vP.vP'$ i.e. $QV^2 : PV.VP'$ is constant.



To obtain the value of this constant, take V at C, and let D be the position of Q.

Then $QV^2 = CD^2$ and $PV.VP' = PC.CP' = CP^2$.

Hence $QV^2 : PV.VP' :: CD^2 : CP^2$,

the formula required.

But this is not the formula given in books on Geometrical Conics; for in the above formula either P or D is imaginary, since, of two conjugate diameters of a hyperbola, one only meets the curve in real points. Take P real and D imaginary. Then CD^2 is negative, otherwise D would be real. On CD take the point d, such that $Cd^2 = -CD^2$. Then d is real, for Cd^2 is positive.

Then $QV^2 : PV.VP' :: -Cd^2 : CP^2$,

i.e. $QV^2 : PV.P'V :: \quad Cd^2 : CP^2$,

which is the formula given in books on Geometrical Conics, the d here replacing the D of the books.

We may call CD the true and Cd the conventional semi-diameter conjugate to CP.

It is sometimes convenient to employ the symbol D for the conventional point d when the meaning is clear from the context.

Note that the locus of d is the so-called conjugate hyperbola.

The theorems of § 5 and § 6 may also be obtained directly from Carnot's theorem by using the triangle contained by DC, VC, VQ.

If we take the transverse axis of the hyperbola as $P'CP$, the relation becomes $QV^2 : AV . A'V :: CB^2 : CA^2$, QV being now perpendicular to the axis $A'CA$; or, as it is more commonly written, $PN^2 : AN . A'N :: CB^2 : CA^2$ which is equivalent to $\dfrac{CN^2}{CA^2} + \dfrac{PN^2}{CB^2} = 1$ where CB is the true conjugate semi-axis, or $\dfrac{CN^2}{CA^2} - \dfrac{PN^2}{CB^2} = 1$ if CB is the conventional semi-axis.

This is called the equation of the hyperbola.

6. *Any parabola may be considered as the limit of an ellipse of which the axes are increased indefinitely.*

Take the direction of the axis AA' of the ellipse, and also the vertex A fixed, but let the vertex A' recede indefinitely. The equation of the ellipse, viz.

$$PN^2 : AN . NA' :: CB^2 : CA^2$$

gives
$$PN^2/AN = \frac{CB^2 . NA'}{CA^2}.$$

But $\dfrac{NA'}{AA'} = \dfrac{AA' - AN}{AA'} = 1 - \dfrac{AN}{AA'} = 1$ when AA' becomes infinite if we keep N fixed. Hence PN^2/AN is ultimately equal to $\dfrac{CB^2 . AA'}{CA^2} = \dfrac{CB^2 . 2CA}{CA^2} = \dfrac{2CB^2}{CA}$ which is the same for all points on the curve. Hence the ellipse becomes a parabola. Also by keeping the value of $2CB^2/CA$ always equal to the required constant value PN^2/AN we can make the ellipse become any given parabola.

Notice that the axis of the parabola is the major axis of the ellipse. For since $2CB^2/CA$ is to be finite when CB and CA are infinite, CA/CB is proportional to CB, i.e. is infinite. Hence $CA > CB$.

Similarly a hyperbola becomes a parabola when one vertex goes to infinity.

7. *The projection of a conic is a conic.*

For since Carnot's theorem is a projective theorem, it holds in the projection of the conic since it holds in the conic. Hence, exactly as in the previous articles, the equation of the projection of the conic can be reduced to the same form as that of a parabola, ellipse or hyperbola. Hence the projection of a conic is a conic. (See also XI. 11.)

8. *One and only one conic can be drawn through five given points.*

Let the points be A, B, C, D, E. Referring to the figure of III. 16, project two vertices U, V of the harmonic triangle of the quadrangle $ABCD$ to infinity and UWV into a right angle. Then $A'B'C'D'$ is a rectangle since UAV is also projected into a right angle. Also $W'U'$ and $W'V'$ are parallel to the sides of the rectangle and half-way between them. Now take $W'U'$ and $W'V'$ as the axes of a conic. Any point on the conic has its CN and PN connected by the relation $\frac{CN^2}{AC^2} + \frac{PN^2}{BC^2} = 1$. But the CN and PN of two points, say, of A' and E', are known, since the positions of A' and E' are known. Hence in the above equation AC and BC are known; i.e. the conic is known. Also B', C', D' will lie on this conic; for D' is the reflexion of A' in $W'U'$, C' of D' in $W'V'$ and B' of C' in $W'U'$. Hence in the new figure A', B', C', D', E' lie on a conic. But the projection of a conic is a conic. Hence in the original figure A, B, C, D, E lie on a conic. Also the above solution is unique; hence only one conic can be drawn through the five points. (See also XI. 6.)

Notice that we have above solved the problem—*Project any conic into a conic and any pair of conjugate lines into the axes.*

We can project the conic into an ellipse by taking W inside and into a hyperbola by taking W outside.

Ex. 1. *M, N are a fixed pair of conjugate points for a conic*

on which moves the variable point O. OM, ON meet the curve again at P, Q. Show that PQ passes through a fixed point.

Take L, the pole of MN; and project LM and LN into the axes.

Ex. 2. *Through B, a vertex of the triangle ABC which is self-conjugate for a conic, is drawn $BPNP'$ meeting the conic in P, P' and AC in N. Any straight line through P meets CA in G and CP' meets PG in O and AB in M.* Prove that

$$\frac{CG \cdot AN}{GN \cdot CA} = 2 \cdot \frac{P'M \cdot OC}{MC \cdot P'O}.$$

Notice that the given relation is projective.

9. *If, on the sides BC, CA, AB of a triangle, the pairs of points A_1A_2, B_1B_2, C_1C_2 be taken, such that*

$$AC_1 \cdot AC_2 \cdot BA_1 \cdot BA_2 \cdot CB_1 \cdot CB_2 = AB_1 \cdot AB_2 \cdot BC_1 \cdot BC_2 \cdot CA_1 \cdot CA_2,$$

then the six points A_1, A_2, B_1, B_2, C_1, C_2 lie on a conic.

Through the five points A_1, A_2, B_1, B_2, C_1 draw a conic. If this conic does not pass through C_2, let AB cut the conic again in C_2'. Then we have by Carnot's theorem

$$AC_1 \cdot AC_2' \cdot BA_1 \cdot BA_2 \cdot CB_1 \cdot CB_2 = AB_1 \cdot AB_2 \cdot BC_1 \cdot BC_2' \cdot CA_1 \cdot CA_2.$$

Dividing the given relation by this relation we have

$$AC_2 / AC_2' = BC_2 / BC_2'.$$

Hence C_2 and C_2' coincide. Hence the six points A_1, A_2, B_1, B_2, C_1, C_2 lie on a conic.

Ex. 1. *The parallels through any point to the sides of a triangle meet the sides in six points on a conic.*

Ex. 2. *A conic can be drawn to touch the three sides of a triangle at their middle points.*

10. *If a chord QQ' of a hyperbola drawn in a fixed direction cuts one of the asymptotes at R, then $RQ \cdot RQ'$ is constant and the same whichever asymptote is taken.*

In the figure of § 5 let Ω be the point at infinity on the asymptote CR. Draw rqq' parallel to RQQ'. Then by Newton's theorem $RQ \cdot RQ' : rq \cdot rq' :: R\Omega^2 : r\Omega^2$. But $R\Omega = r\Omega$. Hence $RQ \cdot RQ' = rq \cdot rq'$; i.e. $RQ \cdot RQ'$ is constant.

Let RQQ' cut the other asymptote at R', then, since $RQ = Q'R'$, it follows that $RQ . RQ' = R'Q . R'Q'$.

It further follows that $RQ . RQ' = RQ . QR' = R'Q' . Q'R$.

The conventional diameter in the direction of an imaginary diameter of a hyperbola is equal to the length intercepted between the asymptotes by a parallel tangent.

For let the tangent TPT' be parallel to the diameter dCd'. Draw RQQ' parallel to each. Then

$RQ . RQ' = TP . TP = TP^2$ (taking R at T)

$= CD . CD' = -CD^2 = Cd^2$ (taking R at C).

Hence $Cd = TP$.

In exactly the same way we prove that *if RQQ' is parallel to the real diameter $P'CP$, then $RQ . RQ' = -CP^2$.*

Notice that *$CPTd$ is a parallelogram,* since Cd is equal and parallel to PT; and so are $CPT'd'$, $CP't'd$, $CP'td'$ and $TT'tt'$.

Ex. 1. *Given a pair of conjugate diameters of a hyperbola, in magnitude and position, to construct the asymptotes and the positions and magnitudes of the axes.*

Drawing parallels through d and d' to CP and a parallel through P to Cd, we get the asymptotes CT and CT'. Now draw RPR' parallel to either axis, then $RP . R'P$ gives the square of this axis.

Ex. 2. *The tangent at Q to a hyperbola meets a diameter CD or Cd (which meets the curve in imaginary points) in T, and the parallel through Q to the conjugate diameter CP meets CD in V; show that $CV . CT = CD^2 = -Cd^2$.*

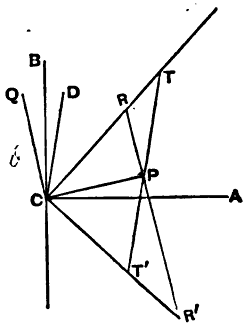
For (DD', TV) is harmonic, and C bisects DD'. Since $CV . CT$ is negative, V and T are on opposite sides of C. Of the above harmonic range notice that DD' are imaginary points and TV real points.

Ex. 3. *Given a pair of conjugate diameters of a hyperbola in position and a tangent and its point of contact, construct the axes in magnitude and position.*

$CV . CT = CD^2$ gives the lengths of the diameters.

11. *In a rectangular hyperbola, conjugate diameters are equally inclined to the asymptotes; and, conversely, if in a hyperbola a pair of conjugate diameters is equally inclined to either asymptote the hyperbola is a rectangular hyperbola.*

For each pair of conjugate diameters is harmonic with the asymptotes; and these are perpendicular, hence the conjugate diameters are equally inclined to them. Conversely, if a pair of conjugate diameters is equally inclined to one asymptote, it must be equally inclined to the other, since the four lines are harmonic. Hence the asymptotes are perpendicular.



In a rectangular hyperbola, conjugate diameters are (*conventionally*) *equal; and, conversely, if two conjugate diameters of a conic are* (*conventionally*) *equal, the conic is a rectangular hyperbola.*

Let CP and CD be the conjugate diameters. Draw the tangent TPT' at P. This is parallel to CD. Hence $CD = TP$. Now $TP = PT'$ and TCT' is a right angle. Hence P is the centre of the circle TCT'. Hence $CP = PT = CD$.

Conversely if $CP = CD$, since $CD = PT$, we have

$$CP = PT = PT'.$$

Hence TCT' is a right angle. Hence the conic is a r. h.

In a rectangular hyperbola perpendicular diameters are (*conventionally*) *equal; and, conversely, if in a conic two perpendicular diameters are* (*conventionally*) *equal, the conic is a rectangular hyperbola.*

Let CP and CQ be the perpendicular diameters. Draw RPR' perpendicular to CP. Then $RP \,.\, PR' = CQ^2$. But since the angles at C and P are right angles, we have $RP \,.\, PR' = CP^2$. Hence $CP^2 = CQ^2$.

Conversely, if $CP = CQ$, then $RP \,.\, PR' = CQ^2 = CP^2$; and RPC is a right angle, hence by Elementary Geometry RCR' is a right angle. Hence the conic is a r. h.

Notice the real relation is in the first case $CP^2 = -CD^2$ and in the second case $CP^2 = -CQ^2$.

If the perpendicular chords LM, $L'M'$ of a r. h. meet at U, then $UL \,.\, UM = -UL' \,.\, UM'$.

For $UL.UM:UL'.UM'$ is the ratio of the squares of the actual parallel diameters. These are perpendicular. Hence the squares are in the ratio -1.

12. *Every rectangular hyperbola which circumscribes a triangle passes through the orthocentre.*

Let ABC be the triangle and H its orthocentre. Suppose a r. h. through ABC cuts the perpendicular AD in Q. Then from the r. h. we have $DQ.DA = -DB.DC$. And from Elementary Geometry we have $DH.DA = -DB.DC$. Hence $DQ = DH$, i.e. Q coincides with H, i.e. the r. h. passes through the orthocentre.

Conversely *every conic which circumscribes a triangle and also passes through the orthocentre of the triangle is a rectangular hyperbola.*

For since $DH.DA = -DB.DC$, the squares of the diameters parallel to DA and BC are equal but have opposite signs. And these diameters are perpendicular. Hence the conic is a r. h.

Hence *through four given points can be drawn one and only one rectangular hyperbola,* viz. the conic through the four points and the orthocentre of any three.

Ex. 1. *If a triangle PQR which is right-angled at Q be inscribed in a r. h., the tangent at Q is the perpendicular from Q on PR.*

For H is also at Q.

Ex. 2. *If a r. h. circumscribe a triangle, the triangle formed by the feet of the perpendiculars from the vertices on the opposite sides is self-conjugate for the r. h.*

Being the harmonic triangle of $ABCP$.

VI

1. Through a point P on a conic chords PA, PB are drawn and chords CC', DD' are drawn parallel to these through the point M, not on the conic. On PA, PB lengths PQ, PR are taken which are inversely proportional to $MC.MC'/PA$ and $MD.MD'/PB$. Prove that the normal at P to the conic is a diameter of the circle PQR.

2. Through a variable point U a chord PQ of a parabola is drawn in a fixed direction and also UR parallel to the axis to meet the parabola at R. Show that $UP \cdot UQ/UR$ is constant.

3. Show that the above theorem does not hold for a hyperbola when UR is parallel to an asymptote unless U lies on a fixed line parallel to an asymptote.

4. A, B, C are three points on a conic. The tangents at A, B, C meet at G, H, K. Points D, E, F are taken on BC, CA, AB such that AD, BE, CF are concurrent. Show that GD, HE, KF are concurrent.

5. $A_1 A_2$ is any chord of a conic parallel to the tangent at A. A line through A cuts the conic at C_1, the chord $A_1 A_2$ at B, and the circle of curvature of A at C'. Show that $AC' = AC_1 \cdot BA_1 \cdot BA_2/BC_1 \cdot BA$.

6. Show that in the figure of § 5 the polar of d is $d'T'$.

7. Every rectangular hyperbola which passes through the middle points of the sides of a triangle passes through the circumcentre.

8. A, B, C are any three points on a circle and PQ is any diameter of the circle. Show that the centres of the rectangular hyperbolas $BCPQ$, $CAPQ$, $ABPQ$ lie on the nine-point circle of ABC.

CHAPTER VII

FOCI OF A CONIC

1. A *focus* of a conic is a point at which every two conjugate lines are perpendicular.

A *directrix* of a conic is the polar of one of the foci. The polar of a focus is called the corresponding directrix.

From the definition of a focus it at once follows that *every two perpendicular lines through a focus are conjugate.*

Ex. 1. *Tangents at the ends of a focal chord meet on the directrix.*

Ex. 2. *If the line joining any point T to the centre meet the directrix in Z, then SZ is perpendicular to the polar of T.*

Being perpendicular to the polar of Z.

***2.** *In the case of a central conic every real focus must lie on the same principal axis.*

For let S be a focus and C the centre. Through S and C draw perpendiculars to SC meeting at infinity at Ω. Then SC and $S\Omega$ are conjugate, being perpendicular lines through a focus; hence the pole of SC is on $S\Omega$. Also, since SC passes through C, its pole is at infinity. Hence the pole of SC is Ω. Hence CS and $C\Omega$ are conjugate diameters; and they are perpendicular, hence they are the axes. Hence each focus lies on one of the axes.

Also we cannot have real foci on each axis. For let S and H be two foci. Draw ΩS and ΩH perpendicular to SH. Then, as above, the pole of SH lies both on $S\Omega$ and on $H\Omega$ and is therefore Ω. Hence SH is the polar of a point at infinity and therefore passes through the centre. Hence S and H lie on the same axis.

Notice that, generally, one and only one pair of conjugate lines at a point is orthogonal. For conjugate lines at a point

are harmonic with the tangents from the point. Hence, if they are also perpendicular, they must be the bisectors of the angles between the tangents.

If two pairs of conjugate lines at a point are orthogonal, every pair is orthogonal, i.e. the point is a focus.

Let O be the point. With O as centre describe a circle. Then the two pairs of orthogonal conjugate lines are conjugate diameters of this circle and hence are harmonic with the tangents of the circle from O. But the tangents of the conic from O are also harmonic with these orthogonal pairs. Hence the two pairs of tangents coincide; for to find a pair of lines harmonic with two pairs of lines is a problem having one and only one solution. Hence the other pairs of conjugate lines are harmonic with the tangents to the circle from O; and hence are orthogonal.

Notice that *a focus of a conic is an internal point*; for we have proved that the tangents to a conic from a focus S are the tangents from S to a circle with centre at S and are therefore imaginary.

If S is a focus of a conic, the tangents from any point subtend equal angles at S; and, conversely, if the tangents TP and TP' of a conic subtend equal angles at the point S, then the line through S conjugate to ST is perpendicular to ST.

In the figure of § 3, draw SK perpendicular to ST to meet PP' at K; and let ST cut PP' at R. Then, if S is a focus, ST and SK, being perpendicular, are conjugate lines; hence the pole of ST lies on SK. But ST passes through T, the pole of PP'; hence its pole lies on PP' and is therefore K. Hence (KR, PP') is harmonic. Hence $S(KR, PP')$ is harmonic. And KST is a right angle. Hence ST bisects the angle PSP'.

Again suppose S is a point such that ST bisects the angle PSP'. Draw SK perpendicular to ST to meet PP' at K. Then $S(KR, PP')$ is harmonic. Hence (KR, PP') is harmonic. Hence the polar of K passes through R; and also through T since K is on PP'. Hence ST is the polar of K. Hence SK and ST are conjugate lines.

An ellipse has two and only two foci.

We know that the foci lie on an axis. Let S be a focus on the axis ACA'; and let BCB' be the other axis. Draw the tangents at A and B meeting at T. Then we know that the tangents TA, TB subtend equal angles at S. Hence $BST = AST = BTS$ by parallels. Hence $BS = BT = CA$. Hence, if foci lie on ACA', they are the intersections of AA' with a circle, centre B and radius CA. Also these intersections are foci. For $AST = BTS = BST$ since $BT = AC = BS$. Hence the line through S conjugate to ST is perpendicular to ST. Also the line conjugate to SA is perpendicular to SA, being the line joining S to the pole of AA'. Hence two pairs of conjugate lines at S are orthogonal. Hence S is a focus.

Notice that the foci are on the major axis; for we must have $BS > BC$ if the circle is to cut AA'; i.e. $CA > CB$.

If, however, $CA = CB$, i.e. *if the ellipse is a circle, both foci coincide at C*, for now $BS = BC$.

A hyperbola has two and only two foci.

In the case of the hyperbola, the foci must lie on the transverse axis; for all points on the conjugate axis are outside the conic. Let S be such a focus. Let the tangent at A and an asymptote meet at T. Then ST bisects the angle $AS\Omega$, Ω being the point at infinity on the asymptote; for the asymptote is the tangent at Ω. Hence

$$CST = TS\Omega = CTS$$

by parallels. Hence $CS = CT$; which determines two and only two foci, for the converse can be proved as in the case of the ellipse.

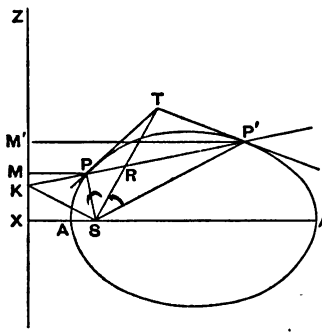
We discuss the foci of a parabola by considering it as the limit of an ellipse when the vertex A' of the major axis AA' goes to infinity, A remaining fixed and CB^2/CA remaining finite. We know that the major axis AA' of the ellipse becomes the axis of the parabola. Hence the foci of the parabola lie on the axis. Again $AS\,[= CA - CS = a - ae = a(1-e^2)/(1+e) = b^2/a(1+e)]$ remains finite; for b^2/a remains finite by hypothesis and e is less than unity.

Hence one focus is at a finite distance. But $AS' > AC$ in the ellipse; hence S' is at infinity in the parabola.

For a complete theory of the foci, real and imaginary, see Chapter XXVIII.

3. *If from any point P on a conic, a perpendicular PM be drawn to the directrix which corresponds to a focus S, then $SP \div PM$ is constant.*

Take any two points P and P' on the conic. Let the tangents at P and P' meet in T. Let PP' meet the corresponding directrix in K, and ST in R. From P and P' drop the perpendiculars PM and $P'M'$ on the directrix.



Now SK and ST are conjugate lines at the focus; for the polar of K, which lies on PP' and on the directrix, is TS. Hence SK is perpendicular to ST. Also $(KPRP')$ is harmonic, since K is the pole of ST. Hence $S(KPRP')$ is harmonic. Hence SK and ST, being perpendicular, are the bisectors of the angle PSP'. Now since SK bisects the angle PSP' (externally in the figure), we have

$$SP : SP' :: PK : P'K :: PM : P'M'. \quad \text{Hence}$$

$SP : PM :: SP' : P'M'$; in other words, $SP : PM$ is constant.

In the parabola, $SP = PM$.

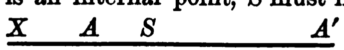
For let SA be the axis. Then SA meets the parabola again at infinity, at Ω, say. Hence $(XAS\Omega)$ is harmonic, since XZ is the polar of S. Hence $SA = AX$.

But $SP : PM :: SA : AX$, for A is on the parabola.

Hence $SP = PM$.

In the ellipse, $SP < PM$.

Since a focus is an internal point, S must lie between A and A'.



Let A be the vertex between S and X. Then since A' is a point on the ellipse, we have $SP : PM :: SA' : A'X$.

But $SA' < A'X$, hence $SP < PM$.

In the hyperbola, $SP > PM$.

Since the focus is an internal point, S must lie outside the segment AA'. $S \quad A \quad X \qquad A'$

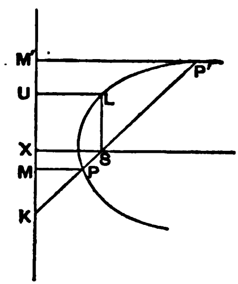
As before $SP : PM :: SA' : A'X > 1$.

The corresponding property in the circle is that the radius is constant.

For the focus is the centre. Hence the directrix is the line at infinity. Hence $PM = P'M'$. Hence $SP = SP'$, i.e. $CP = CP'$.

4. We have now shown that every conic has a focus and that this focus possesses the $SP : PM$ property by which a focus is defined in books on Geometrical Conics. This opens up to us all the proofs given in such books. It will be assumed that these proofs are known to the reader; and the results will be quoted when convenient. Properties of Conics which can be best treated by the methods of Geometrical Conics will be usually omitted from this treatise.

5. *In any conic, the semi-latus rectum is equal to the harmonic mean between the segments of any focal chord.*



Let the focal chord $P'SP$ cut the directrix in K.

Then $(KPSP')$ is harmonic since S is the pole of XK. Hence

$$2(KS)^{-1} = (KP)^{-1} + (KP')^{-1}.$$

But

$$KP : KS : KP' :: PM : SX : P'M'$$
$$:: SP : SL : SP',$$

for $SP : PM :: SL : LU :: SL : SX$.

Hence

$$2(SL)^{-1} = (SP)^{-1} + (SP')^{-1}.$$

Ex. 1. *If T be the pole of the focal chord PQ of a parabola, show that* $PQ \propto ST^2$.

For $PQ \propto SP \, . \, SQ$.

Ex. 2. *A focal chord of a central conic is proportional to the square of the parallel diameter.*

***6.** *If the tangent at P meet the tangents at the vertices AA' of the focal axis in UU', then UU' subtends a right angle at S and S'. Also if US, $U'S'$ cut in E, and US', $U'S$ cut in F, then EF is the normal at P.*

For since AU and PU subtend equal angles at S and since $A'U'$ and PU' subtend equal angles at S, it follows that USU' is a right angle. Similarly UU' subtends a right angle at S'.

Again, F is the orthocentre of the triangle UEU'. Hence EF is at right angles to UU'. Let PU cut the axis in T and draw the ordinate PN. Then $(TUPU') = (TANA')$ is harmonic. Also if EF cut UU' in P', then since UU' is a harmonic side of the quadrilateral SF, FS', $S'E$, ES, we have $(TUP'U')$ harmonic. Hence P' and P coincide, i. e. EF passes through P. Hence EF is the normal at P.

Ex. 1. *If a circle through the foci cut the tangent at the vertex A in U, V and the tangent at the vertex A' in U', V', show that the diagonals of the rectangle $UU'V'V$ touch the conic.*

Ex. 2. *Given the focal axis AA' in magnitude and position and one tangent, construct the foci.*

7. *If the tangent at a point P of a central conic cut the focal axis in T, and if the normal at P cut the same axis in G, then $CG \cdot CT = CS^2$.*

For since the tangent and normal bisect the angle SPS', it follows that $P(SS', TG)$ is harmonic; hence

$$CG \cdot CT = CS^2.$$

Ex. 1. *Given the axes in position and one tangent and its point of contact, construct the foci.*

Ex. 2. *In the parabola, S bisects GT.*

For S' is at infinity.

Ex. 3. *Given the axis of a parabola in position and one tangent and its point of contact, construct the focus.*

Confocal Conics.

8. Confocal conics (or briefly confocals) are conics which have the same foci. If one of the given foci is at infinity, we have confocal parabolas, which may also be defined as parabolas having the same focus and the same axis.

Two confocals can be drawn through any point, one an ellipse and one a hyperbola, and these cut at right angles.

Join the given point P to the foci S, S', and draw the bisectors PL and PL' of the angle SPS'. Since both foci are finite, the conic must be an ellipse or a hyperbola. If it be an ellipse, then Q being any point on the ellipse,

$$SQ + S'Q = SP + S'P;$$

so that one and only one ellipse can be drawn through P with S and S' as foci. Similarly one and only one hyperbola can be drawn. And the two conics cut at right angles, for PL and PL' are their tangents at P.

If one focus is at infinity, the ellipses and hyperbolas become parabolas, and we get the theorem—

Of the system of parabolas which have the same focus and the same axis, two pass through any point and these are orthogonal.

This can be easily proved directly.

One confocal and one only can be drawn to touch a given line.

Take R, the reflexion of S in the tangent t. Then if $S'R$ cuts t at P, t bisects the angle SPS'. Two cases arise.

(i) If t bisects the angle SPS' externally, t will touch the ellipse described with foci S and S' and major axis equal to $SP + S'P$.

(ii) If t bisects SPS' internally, t will touch the hyperbola with S, S' as foci and $SP \sim S'P$ as transverse axis.

If one focus is at infinity we get the theorem—

Of a system of confocal parabolas, one and one only touches a given line.

This can be easily proved directly.

9. *The locus of the poles of a given line for a system of confocals is a line.*

Let the given line be LM, and let V be the point of

contact of the confocal which touches LM. Draw VL' perpendicular to VL. Then VL' contains the pole of LM for any confocal.

Since V is the pole of LM for the confocal which touches LM, the pole of LM for this confocal is on VL'. From V draw the tangents VT and VT' to any other confocal. Now VL and VL' bisect SVS', for they are the tangent and normal to the confocal touching LM. Also $\angle TVS = \angle T'VS'$ by VIII. 18. Hence VL and VL' are the bisectors of TVT', i.e. VL and VL' are harmonic with VT and VT'. Hence VL, VL' are conjugate for this confocal, i.e. for any confocal of the system. Hence the pole of VL for any confocal lies on VL'.

The theorem follows for the confocals to which real tangents cannot be drawn from V by the principle of continuity.

We have incidentally proved the proposition—

If V be any point in the plane of a conic whose foci are S and S', then the bisectors of the angle SVS' are conjugate for the conic.

If one focus is at infinity, we get the theorem—

The locus of the poles of a given line for a system of confocal parabolas is a line.

If V be any point in the plane of a parabola whose focus is S, and if VM be parallel to the axis, the bisectors of the angle SVM are conjugate for the parabola.

Ex. 1. *If a triangle be inscribed in one conic and circumscribed to a confocal, the points of contact are the points of contact of the escribed circles.*

Let ABC be the triangle. Let the tangents at A and B meet in R. Then the locus of the poles of AB is the normal at the point of contact N of AB, i.e. RN is perpendicular to AB. And R is the centre of the escribed circle because the external angles at A and B are bisected.

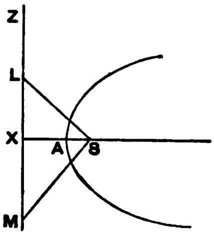
Ex. 2. *From T are drawn the tangents TP, TP' to a conic and the tangents TQ, TQ' to a confocal; show that the angle QPQ' is bisected by the normal at P.*

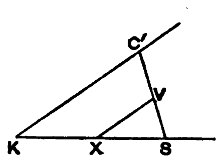
For the normal at P meets QQ' in the pole of TP for the other conic.

Focal Projection.

***10.** *To project a given conic into a circle so that a focus of the conic may be projected into the centre of the circle; and to show that angles at the focus are projected into equal angles at the centre.*

Let S be the focus to be projected into the centre of the circle; and let XZ be the corresponding directrix. Since S is to be projected into the centre, its polar XZ must be projected to infinity. Rotate S about XZ into any position out of the plane of the conic, and take this position as the position of the vertex of projection V. With V as vertex project the conic on to a plane parallel to VXZ. Now the projection of a conic is a conic. Also C', the projection of S, is the centre of the new conic; for the polar of S is projected to infinity, hence C' is the pole of the line at infinity. Again, the angle LSM at S is superposable to the angle LVM; and the projection of SL is parallel to VL, and the projection of SM to VM. Hence LSM is projected into an equal angle at C'; so every angle at S is projected into an equal angle at C'. Also conjugate lines at S are projected into conjugate lines at C'. Hence the perpendicular conjugate lines at S are projected into perpendicular conjugate lines at C', i.e. every two conjugate lines through the centre C' are perpendicular. Hence the new conic is a circle by V. 14.





Ex. 1. *Project a conic into a conic so that one focus of the one shall project into one focus of the other, and any line shall be projected to infinity.*

Ex. 2. *Project a circle into a conic so that the centre of the circle shall project into a focus of the conic.*

VII

1. PSQ is a focal chord of a conic. UOV is any chord of the conic through the middle point O of PQ. Parallels through U, V to PQ meet the directrix corresponding to S at M, N. Show that PQ bisects the angle MSN.

2. Two parabolas have a common focus S and are such that the vertex of one is an extremity of the latus rectum of the other. The line joining their other real point of intersection K to the focus meets the parabolas again at P, Q. Show that $QPSK$ is a harmonic range.

3. The length of the tangent at a point P of a hyperbola between P and AA' is a harmonic mean between the perpendiculars from S and S' on the normal at P.

CHAPTER VIII

RECIPROCATION

1. If we have any figure determined by points A, B, C, ... and lines l, m, n, ..., we can form another figure called a *reciprocal figure* in the following way. Choose any conic Γ called the *base conic*. Take the polar a of A for this conic, the polar b of B, the polar c of C, ...; also take the pole L of l for this conic, the pole M of m, the pole N of n, ...; then the figure determined by the lines a, b, c, ... and the points L, M, N, ... is said to be reciprocal to the figure determined by the points A, B, C, ... and the lines l, m, n, ...; also the point A and the line a are said to be *reciprocal*, so also B and b, C and c, ..., l and L, m and M, n and N, ...

The name reciprocal arises from the following property—

If the reciprocal of the figure α be the figure α', then the reciprocal of α' is α.

For let A be a point of the figure α. The reciprocal of A is the polar a of A for the base conic Γ. Hence a is one of the lines of α' the reciprocal of α. Again, in obtaining α'', the reciprocal of α', we should obtain the pole of a (a line of α') for Γ; but the pole of a is A. Hence A is a point in α''. Hence every point belonging to α belongs also to α''. So every line belonging to α belongs also to α''. Hence α and α'' coincide.

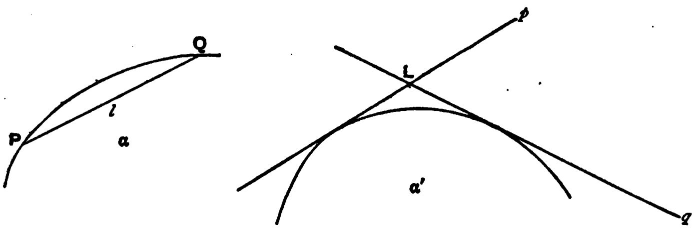
The reciprocal of the join of two points A, B is the meet of the reciprocal lines a, b; and the reciprocal of the meet of two lines l, m is the join of the reciprocal points L, M.

By definition the reciprocal of AB is the pole of AB for the base conic Γ. But the pole of AB is the meet of the polars of A and B for Γ, i.e. is the meet of the reciprocal lines a and b. Similarly the second part follows.

2. A curve may be considered either as the locus of points

on it or as the envelope of tangents to it. Hence the *reciprocal of a curve* may be defined either as the envelope of the polars for the base conic Γ of points on the given curve or as the locus of the poles for Γ of the tangents to the given curve. These definitions determine the same curve.

For take two points P and Q on the given curve a and the polars p and q of P and Q for the base conic Γ. Then by the first definition p and q touch the reciprocal curve a' of a. Now the reciprocal of l, the join of P and Q in a, is the meet L of p and q in a'. Also when P and Q coincide, PQ becomes a tangent to a. At the same time p and q coincide and L



becomes a point on a'. Hence the reciprocal of a tangent to a is a point on a'. Which agrees with the second definition.

From the above we see that—*the reciprocals of a point P on a curve and the tangent l to the curve at P are a tangent p to the reciprocal curve and the point of contact L of p.*

The reciprocal of a point of intersection of two curves is a common tangent to the reciprocal curves.

For let l and m be the tangents to the curves a and β at their meet P. In the reciprocal figure we shall have two curves a' and β' which have one tangent p with different points of contact L and M.

The reciprocal of two curves touching is two curves touching.

For the reciprocal of l touching both a and β at P is L, the point of contact of p with both a' and β'.

Ex. 1. *The reciprocal of a conic, taking the conic itself as base conic, is the conic itself.*

Ex. 2. *The reciprocal of a circle, taking a concentric circle as base conic, is a circle concentric with both.*

3. *Whatever base conic is taken, the reciprocal of a conic is a conic.*

From any point can be drawn two tangents real or imaginary to the given conic. Hence every line meets the reciprocal curve in two points real or imaginary; hence the reciprocal curve is a conic. (For another proof see XIII. 2.)

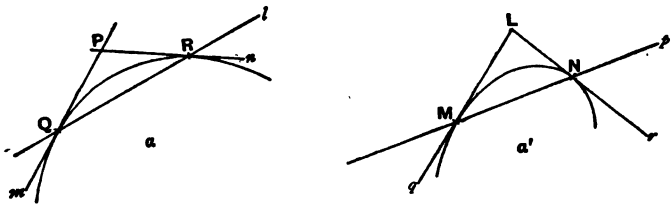
More generally. *If the degree of a curve is m and its class n, then the class of the reciprocal curve is m and its degree is n.*

For a line cuts the given curve in m points; hence from any point can be drawn m tangents to the reciprocal curve. Also from any point can be drawn n tangents to the given curve; hence any line cuts the reciprocal curve in n points.

Ex. 1. *The reciprocal of two conics having double contact is two conics having double contact.*

Ex. 2. *The reciprocal of a common chord of two conics is a meet of common tangents of the reciprocal conics.*

4. *If the point P be the pole of the line l for the conic a and if p, L, a' be the reciprocals of P, l, a for any base conic, then the line p is the polar of the point L for the conic a';* or briefly—*the reciprocal of a pole and polar for any conic is a polar and pole for the reciprocal conic.*



From P draw the real or imaginary tangents m, n to a touching in Q, R. Then QR is l, the polar of P for a. The reciprocals of Q and m in a are a tangent q to a' and its point of contact M; so for r and N. The reciprocal of the meet P of the tangents m and n at Q and R is the join p of the points of contact M and N of the tangents q and r. Again, the reciprocal of l, the join of Q and R, is the meet of q and r, i.e. is L. Hence the reciprocals of P and l which are pole

and polar for a are p and L which are polar and pole for a'. (For another proof see XIII. 3.)

The reciprocals of conjugate points are conjugate lines.

For if the point P is conjugate to the point Q, then the polar l of Q passes through P. Hence in the reciprocal figure the pole L of q lies on p, i.e. the reciprocals p and q of P and Q are conjugate lines. Similarly—

The reciprocals of conjugate lines are conjugate points.

Ex. *The reciprocal of a triangle self-conjugate for a conic is a triangle self-conjugate for the reciprocal conic.*

5. It will be found that all geometrical theorems occur in pairs called *reciprocal theorems.* Thus the theorems (i) '*The harmonic points of a quadrangle inscribed in a circle are the vertices of a triangle self-conjugate for the circle,*' and (ii) '*The harmonic lines of a quadrilateral circumscribed to a circle are the sides of a triangle self-conjugate for the circle,*' are reciprocal theorems. The reason of the name is that each can be derived from the other by reciprocation. Hence we need only have proved half the theorems in the former part of the book; the other half might have been deduced by reciprocation. This method will be often used in future to duplicate a theorem.

For example, to deduce the second of the above theorems from the first, reciprocate, taking the given circle as base conic. The reciprocals of four points on the circle are the polars of these points for the circle, i.e. are the tangents at these points, and so on step by step; and the triangle obtained is self-conjugate because the reciprocal of a self-conjugate triangle is a self-conjugate triangle.

6. If one conic only is involved it is best to reciprocate for this conic itself, as then a theorem about a circle gives a theorem about a circle, a theorem about a parabola gives a theorem about a parabola, and so on. In this way we get a theorem as general as the given one.

7. From any proposition we can derive another proposition by reciprocation. Thus from Pascal's theorem, viz. 'If A,

B, C, D, E, F are any six points on a conic, and if the lines joining AB, BC, CD, DE, EF, FA be called l, m, n, r, s, t and if the point of intersection of l, r be called X, of m, s be called Y, and of n, t be called Z, then X, Y, Z are collinear,' we derive Brianchon's theorem, viz. 'If a, b, c, d, e, f are any six tangents of a conic and if the points of intersection of ab, bc, cd, de, ef, fa be called L, M, N, R, S, T, and if the line joining L, R be called x, the line joining M, S be called y, and the line joining N, T be called z, then x, y, z are concurrent.'

For the reciprocal

of	is
point	line
conic	conic
point on conic	tangent to conic
connector of two points	intersection of two lines
intersection of two lines	connector of two points
points on a line i.e. collinear points.	lines through a point i.e. concurrent lines.

The reader should write down the reciprocals of the following propositions in the same way.

1. If two vertices of a triangle move along fixed lines while the sides pass each through a fixed point, the locus of the third vertex is a conic section.

2. If a triangle be inscribed in a conic, two of whose sides pass through fixed points, the envelope of the third side is a conic, having double contact with the given conic.

3. Given two points on a conic and two tangents, the line joining the points of contact of these tangents passes through one or other of two fixed points.

4. Given four tangents to a conic, the locus of the poles of a fixed line is a line.

5. Given four points on a conic, the locus of the poles of a given line is a conic.

6. Inscribe in a conic a triangle whose sides shall pass through three given points.

7. If three conics have two points common or if they have

each double contact with a fourth, the six meets of common tangents lie three by three on the same lines.

8. The meets of each side of a triangle with the corresponding side of the triangle formed by the polars of the vertices for any conic lie on a line.

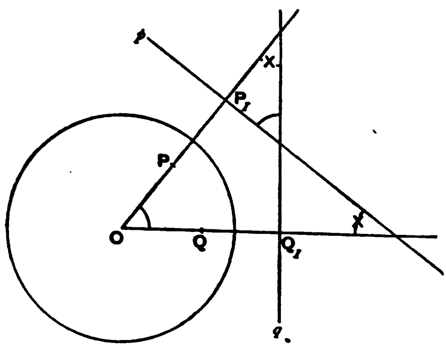
9. If through the point of contact of two conics which touch, any chord be drawn, the tangents at its ends will meet on the common chord of the two conics.

10. If α and β be two conics having each double contact with the conic γ, the chords of contact of α and β with γ and their common chords with each other meet in a point.

11. If α, β, γ be three conics, having each double contact with the conic σ, and if α and β both touch γ, the line joining the points of contact will pass through a meet of the common tangents of α and β.

Point Reciprocation.

8. If the base conic is a circle (the most common case), the reciprocation is generally called *point reciprocation*, the centre O of the base circle is called the *origin of reciprocation*,



and the radius k of the base circle is called the *radius of reciprocation*. The reason of the name point reciprocation is that the value of k is usually of no importance. By reciprocation is meant point reciprocation unless the contrary is stated or implied in the context.

In point reciprocation, the angle between two lines is equal to the angle subtended by the reciprocal points at the origin of reciprocation; or briefly, $\angle POQ = \angle p, q$.

Let p and q be the lines, and P and Q the reciprocals of p and q. Let O be the origin of reciprocation. Then P being the pole of p for a circle whose centre is O, OP is perpendicular to p. So OQ is perpendicular to q. Hence POQ is equal to the angle between p and q.

In point reciprocation, the angle between a line p and the line joining the origin O of reciprocation to a point Q, is equal to the angle between the line q and OP, P and q being the reciprocals of p and Q; or briefly, $\angle OP, q = \angle OQ, p$.

This follows at once, as before, from the above figure.

In point reciprocation, if P be the reciprocal of p and if O be the origin of reciprocation, then OP is inversely proportional to the perpendicular from O on p.

For $\quad OP \,.\, OP_1 = OP \,.\, (O, p) = k^2$.

9. The reciprocal of a figure for a given point O and a given radius k may be obtained without considering a circle at all. To obtain the reciprocal of P—on OP take a point P_1, such that $OP \,.\, OP_1 = k^2$, and through P_1 draw a perpendicular p to OP. To obtain the reciprocal of p—drop the perpendicular OP_1 from O to p, and on OP_1 take the point P, such that

$$OP \,.\, OP_1 = k^2.$$

Instead of taking $OP \,.\, OP_1 = k^2$, we may take

$$OP \,.\, OP_1 = -k^2,$$

i.e. we may take P and P_1 on opposite sides of O. This is called *negative reciprocation*, and is equivalent to reciprocating for an imaginary circle whose radius is $k\sqrt{-1}$.

Notice that *the reciprocal of the origin of reciprocation is the line at infinity; and conversely, the reciprocal of the line at infinity is the origin.*

For the polar of the centre of the base circle is the line at infinity; and conversely.

Also *the reciprocal of a line through the origin is a point at infinity; and conversely.*

Ex. 1. *Reciprocate a quadrangle into a parallelogram.*

Take O at one of the harmonic points.

Ex. 2. *The reciprocal of the meet of OP and m is the line through M parallel to p.*

Ex. 3. *If P and Q be points on a curve such that PQ passes through O, then in the reciprocal for O, p and q are parallel tangents.*

Ex. 4. *The reciprocal for O of the foot of the perpendicular from O on p is the line through P perpendicular to OP.*

Ex. 5. *The reciprocal of a triangle for its orthocentre is a triangle having the same orthocentre.*

Let ABC be the triangle and O the orthocentre. Let the sides BC, CA, AB be called x, y, z. Then the new triangle is XYZ with sides a, b, c. Also OA is perpendicular to x; hence OX is perpendicular to a since $\angle OA, x = \angle OX, a$. So OY is perpendicular to b and OZ to c.

Ex. 6. *On the sides, BC, CA, AB of a triangle are taken points P, Q, R such that the angles POA, QOB, ROC are right, O being a fixed point; show that P, Q, R are collinear.*

Reciprocating for O, we have to prove that the three perpendiculars from the vertices on the opposite sides meet in a point.

Ex. 7. *The reciprocal of the curve $p = f(r)$ for the origin is $k^2/r = f(k^2/p)$.*

Let b be the tangent at A to the given curve. Then B is on the reciprocal curve and a touches it. Hence
$p = (O, b) = k^2/OB = k^2/r'$, and $r = OA = k^2/(O, a) = k^2/p'$.

Reciprocation of a conic into a circle.

10. *The reciprocal of a circle, taking a circle with centre O as base conic, is a conic having a focus at O.*

Let U be the centre of the given circle a. Take u the reciprocal of U, i.e. the polar of U for the base circle Γ whose centre is O. Let p be any tangent to a touching at T. Take P the reciprocal of p. Draw the perpendicular PM from P to u.

Then since p is the polar of P and u the polar of U for Γ, we have by Salmon's theorem (III. 9)

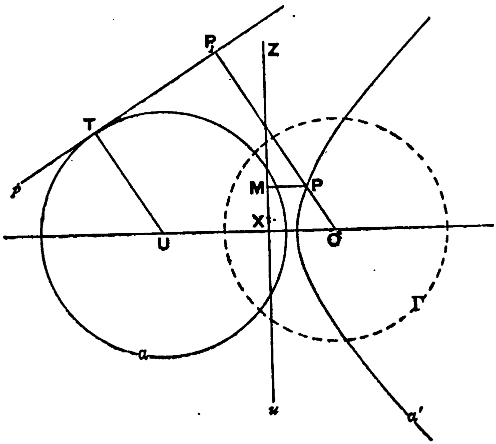
$$OP/(P, u) = OU/(U, p), \quad \text{i.e.} \quad OP/PM = OU/UT.$$

Hence OP/PM is constant, i.e. the locus of P is a conic with O as focus. But the reciprocal of a for Γ is the locus

of the poles for Γ of the tangents to a, i.e. is the locus of P. Hence the reciprocal of a circle a for the circle Γ whose centre is O is a conic of a' having a focus at O.

Briefly, *the reciprocal of a circle for a point O is a conic having a focus at O.*

Since $e = OP/PM = OU/UT$, we see that the reciprocal of a circle for a circle whose centre is O, is an ellipse, parabola or hyperbola according as $OU <=> UT$, i.e.



according as O is inside, on or outside the given circle. This is a particular case of a general theorem. (See § 21.)

Let $OU = \delta$, $UT = R$, and let k be the radius of the base circle. Then $e = \delta/R$. Also $OX \,.\, OU = k^2$.

Hence $k^2/\delta = OX = a/e - ae$. Hence $a = k^2R/(R^2 - \delta^2)$ in the case of the ellipse.

Similarly in the hyperbola $a = k^2R/(\delta^2 - R^2)$.

11. Conversely, *the reciprocal of a conic, taking any circle whose centre is at a focus as base conic, is a circle.*

Let O be the given focus, and XZ or u the corresponding directrix. Take any point P on the conic a', and let p be its reciprocal, i.e. the polar of P for the base circle Γ whose centre is at O. Draw the perpendicular PM from P to u.

Take the reciprocal U of u. Draw the perpendicular UT from U to p.

Then since p is the polar of P and u the polar of U for the conic Γ, we have by Salmon's theorem

$$OU/UT = OP/PM = e.$$

Hence OU/UT is constant. Also U is a fixed point; hence UT is of constant length. Hence the perpendicular from U on p is constant, i.e. p envelopes a fixed circle a. But the reciprocal of a' for Γ is the envelope of the polars for Γ of the points on a'. Hence the reciprocal of the conic a' for a circle Γ whose centre is at one of the foci O of the conic is a circle a.

Briefly, *the reciprocal of a conic for one of its foci is a circle.*

Ex. 1. *The envelope of the polar for a of the centre of a circle which touches two given circles a and β is a circle.*

For $AP - BP$ is constant.

Ex. 2. *Given four points A, B, C, D, show that, with D as focus, one conic can be drawn touching BC, CA, AB, and four conics through ABC. Show also that, if ADB be a right angle, a conic, with focus at D, can be found to touch the five conics.*

Reciprocate for D; and notice that in a right-angled triangle the nine-point circle touches the circumcircle also.

Ex. 3. *Three conics a, β, γ which have a focus in common are such that a touches β in R, β touches γ in P, and γ touches a in Q. Show that the tangents at P, Q, R meet the corresponding directrices of a, β, γ in three collinear points.*

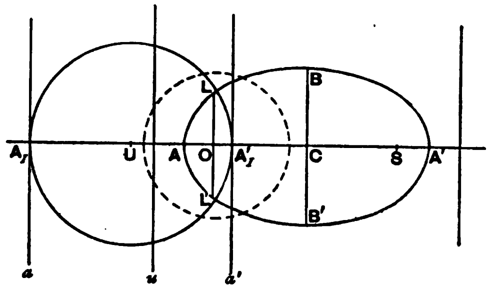
Ex. 4. *Reciprocate the centres of similitude of two circles.*

The two circles reciprocate into conics having a common focus S. Let u, u' be the directrices corresponding to S. Then two common chords pass through the meet of u and u'; and these chords are the reciprocals of the centres of similitude.

Ex. 5. *The reciprocal of two circles for either centre of similitude is two similar and similarly situated conics with a common focus as centre of similitude.*

Reciprocate a pair of parallel tangents.

12. The figures of the reciprocals of an ellipse, a parabola and a hyperbola are given below. In the first figure in each case the curves are in their proper relative positions; the second figure represents the circle separately and the third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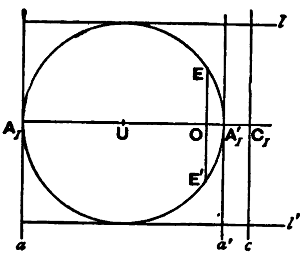
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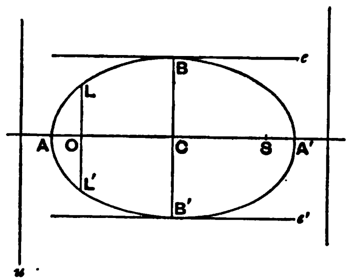
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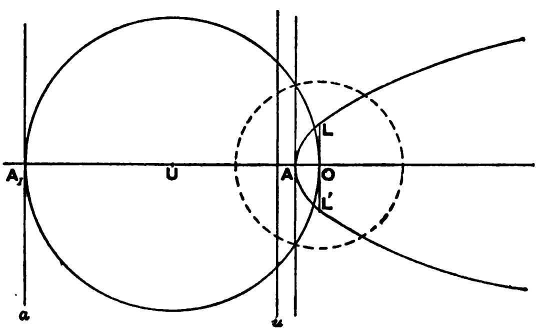
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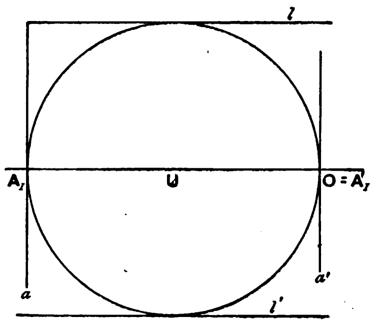
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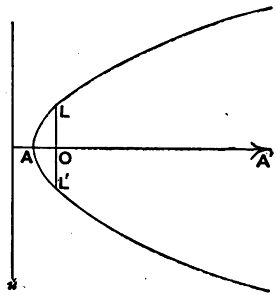
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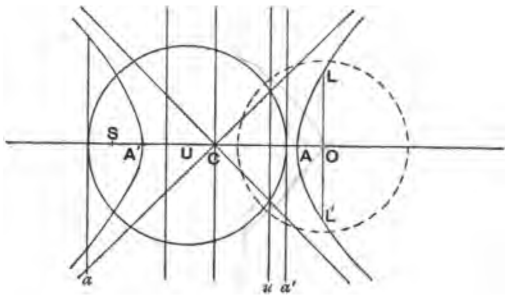
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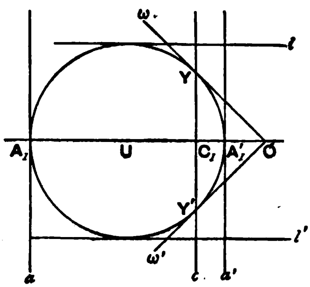
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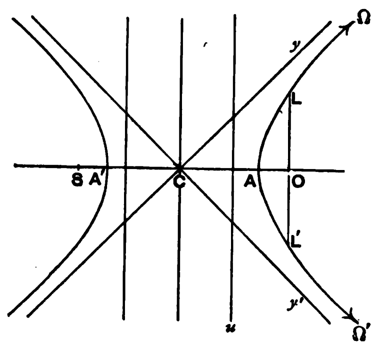
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figure represents the conic separately, so that if one figure be slid on to the other, so that O in one comes on O in the other, we get the proper figure as in the first figure. To avoid complication the figures will generally be separated as in the second and third figures.

13. We already know that the reciprocal of O is the line at infinity and the reciprocal of the *line at infinity* is O. Also that the reciprocal of the directrix u corresponding to O is the centre U of the circle.

The *centre* C of the conic is the pole of the line at infinity for the conic. Hence the reciprocal of the centre is the polar c of O for the circle.

The *asymptotes* y, y' are the tangents from C to the conic. Hence the reciprocals of the asymptotes are the points in which c meets the circle; i.e. the points in which the polar of O for the circle meets the circle.

The reciprocals of the *vertices* A, A' are clearly the tangents at the points where OU meets the circle. In the parabola A' is at infinity; hence its reciprocal is the tangent at O.

The reciprocals of the *vertices* B, B' are clearly the tangents to the circle at E, E', the points where the perpendicular through O to OU meets the circle. Also the reciprocals of E, E' are clearly the tangents at B, B'; it follows that

$$b = CB = k^2/OE.$$

The reciprocals of L, L', *the ends of the latus rectum* LOL', are clearly the tangents l, l' of the circle parallel to OU. Hence $l = OL = k^2/R$. Hence *equal circles reciprocate into conics having equal latera recta.* Notice that l is independent of δ; i.e. of the relative positions of the given circles.

The reciprocal of the second focus S *is the line half-way between* O *and its polar for the circle.*

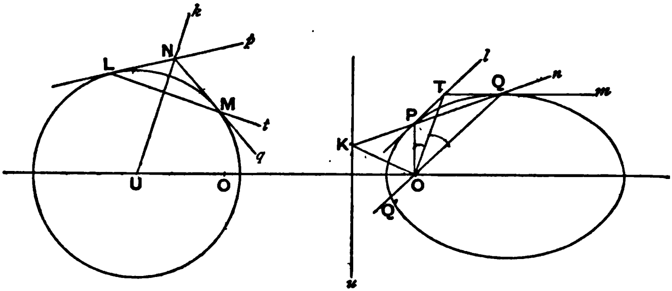
For $OS = 2 . OC$; hence $OC_1 = 2 . OS_1$, where C_1 and S_1 are the points where the reciprocals of C and S meet OU.

Ex. *The reciprocals of coaxal circles for any point on the radical axis are conics having equal minor axes.*

For OE is constant.

14. *If the polar of a point T for a conic meet the conic in P, Q and a directrix in K, then, O being the corresponding focus, the bisectors of the angle POQ are OT and OK.*

Let the two tangents l and m of the conic touch in P and Q and meet in T, and let n be the chord of contact. Let O be a focus of the conic and u the corresponding directrix, and let PQ meet u in K. Then we have to prove that OT and OK are the internal and external bisectors of POQ.



Reciprocate the conic for a circle with centre at O. Then in the reciprocal figure p and q touch the circle at L and M and meet in N, and t is the chord of contact. Also the reciprocal of K, the meet of n and u, is NU.

Now $\angle POT = \angle tp$: so $\angle TOQ = \angle tq$. But $\angle tp = \angle tq$. Hence $\angle POT = \angle TOQ$. Again

$$\angle POK = \angle pk = 180° - \angle qk = 180° - \angle QOK = \angle KOQ',$$

if we produce QO to Q'. Hence OT bisects $\angle POQ$, and OK bisects the supplement $\angle POQ'$.

Note that if TP and TQ had been drawn to touch different branches of a hyperbola, OT would have been the external bisector and OK the internal, instead of as above.

Ex. 1. *XYZ is a triangle circumscribing a conic and P is the point of contact of XY. Show that PX and YZ subtend equal angles at a focus.*

Ex. 2. *If the chord PQ of a conic subtend at the focus O a constant angle, the envelope of PQ is a conic having O as a focus; and the directrices corresponding to O in the two conics coincide.*

For if $\angle POQ$ is constant, then $\angle pq$ is constant; hence the locus of N is a circle having U as centre. Hence the envelope of n is a conic having O as focus and u as corresponding directrix.

Ex. 3. *Find the locus of T when $\angle POQ$ is constant.*

Ex. 4. *From two conjugate points on the directrix of a conic are drawn four tangents to the conic. Show that the locus of each of the other meets of the tangents is a single conic; and that the given directrix is a directrix of this conic, and that the corresponding foci of the two conics coincide.*

Notice that conjugate points on u reciprocate into perpendicular lines through U.

Ex. 5. *The semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord of the conic.*

For P and Q reciprocate into parallel tangents of the circle. Hence $OP^{-1} + OQ^{-1} = k^{-2}(OP_1 + OQ_1) = k^{-2} . 2R = 2l^{-1}$.

Ex. 6. *A pair of parallel tangents to a conic meet a perpendicular to them through a focus in Y and Z and the corresponding directrix in M and N. Show that MZ and NY touch the conic.*

If the tangent MY be called t, then Y reciprocates into the line y through T perpendicular to OT.

Ex. 7. *On the tangent at P to a conic is taken a point Q, such that PQ subtends at a focus S a given angle; show that the locus of Q is a conic having a focus at S. Show also that its eccentricity is to the eccentricity of the given conic as its latus rectum is to the latus rectum of the given conic.*

Ex. 8. *From two points on a directrix are drawn two pairs of conjugate lines. Show that these lines touch a conic with a focus at the corresponding focus.*

Ex. 9. *Reciprocate for any point the theorem—'The locus of the points of contact of tangents from a fixed point to a system of concentric circles is a circle through the fixed point and through the common centre.'*

The reciprocal theorem is:—'If a fixed line a meet at P, Q a variable conic having a given focus, S, and corresponding directrix, u, the envelope of the tangents at P, Q is a conic having a focus at S and touching a and u.'

Ex. 10. *Reciprocate for the centre of the given circle—'The joins of two fixed points on a given circle with the ends of a variable diameter meet at P on a fixed circle through the fixed points and orthogonal to the given circle. Also the tangent at P to the locus is parallel to the diameter.'*

Notice that 'two curves cut orthogonally' reciprocates into 'the angle subtended at O by a common tangent is a right angle.'

Ex. 11. *Reciprocate for any point—'The bisectors of the angles of a triangle meet, three by three, in the centres of the four circles touching the sides.'*

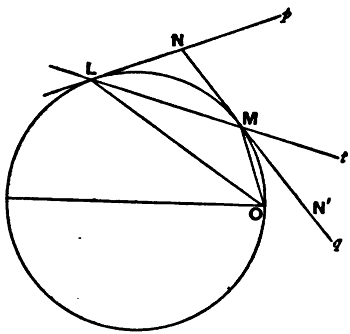
We get 'XYZ is a triangle and O is any point. The bisectors of the angles XOY, YOZ, ZOX meet XY, YZ, ZX at N, N', L, L', M, M'. Then the points L, L', M, M', N, N' lie three by three on four lines which are the directrices of the four conics which can be drawn with a focus at O to pass through X, Y, Z.'

Ex. 12. *Also—'The chord of a circle which subtends a right angle at a fixed point on the circle passes through the centre.'*

Ex. 13. *If a circle be reciprocated into a hyperbola, taking a circle with centre O as base conic, then $BC = k^2/OT$, OT being the tangent from O to the circle.*

For the perpendicular from a focus of a hyperbola on an asymptote is of length BC.

15. *The triangles subtended at the focus of a parabola by any two tangents are similar.*



The reciprocal of the parabola for its focus O is a circle through O.

We have to prove that

$$\angle PTO = \angle TQO \text{ and } \angle POT = \angle TOQ.$$

Now $\angle PTO$, the angle between the line l and the radius OT, is equal to the angle between the radius OL and the line t, i.e. equals $\angle OLM$. So $\angle TQO$ is equal to the angle between

OM and q, i.e. equals $\angle OMN'$. But $\angle OLM = \angle OMN'$. Hence $\angle PTO = \angle TQO$. As before, $\angle POT = \angle TOQ$ follows from $\angle NLM = \angle NML$.

Ex. 1. *Obtain a property of a circle from the theorem—'The orthocentre of a triangle circumscribing a parabola is on the directrix.'*

Ex. 2. *Reciprocate for O the theorem—'If from any point O on a circle perpendiculars be drawn to the sides of an inscribed triangle, the feet lie on a line.'*

We get—'If O be the focus of a parabola and PQR the vertices of a circumscribed triangle, then the perpendiculars through P, Q, R to OP, OQ, OR meet in a point.'

Ex. 3. *The circle which circumscribes a triangle whose sides touch a parabola passes through the focus.*

For the points A, B, C, O lie on a circle if the angles B and O are supplementary.

Ex. 4. *Find by reciprocation the locus of the meet of tangents to a parabola which cut* (i) *at a given angle,* (ii) *at right angles.*

In the figure of the text $\angle LOM$ is given. Hence the envelope of LM is a concentric circle (or the centre, if $LOM = 90°$). Hence the locus is a conic having one focus and the corresponding directrix in common with the parabola (or the directrix, if the tangents are at right angles).

16. *Find the envelope of a chord of a circle which is bisected by a given line.*

Let the chord p of the circle be bisected by the fixed line l in the point Q. Take O the centre of the circle; then OQ is perpendicular to p. Reciprocate for the circle itself. Then P is the foot of the perpendicular from O on the variable line q through the fixed point L. Hence the locus of P is a circle on OL as diameter, i.e. a circle through O and having the opposite point at L. Hence the required envelope is a parabola with focus at O and having its vertex at L_1 the foot of the perpendicular from O on l. Hence the envelope is completely determined.

Ex. 1. *A, B, C, D are four points on a circle, and AC, BD meet at right angles at a fixed point; show that AB, BC, CD, DA envelope one and the same conic.*

Let AC, BD meet in O. Reciprocate for O and we obtain the property of the director circle.

Ex. 2. *The envelope of the base BC of a triangle ABC whose vertex A and vertical angle BAC are given and whose base angles move on fixed lines is a conic one of whose foci is A.*

Reciprocate for A.

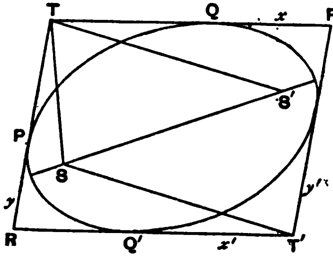
Ex. 3. *Find the envelope of the asymptotes of a system of hyperbolas having the same focus and corresponding directrix.*

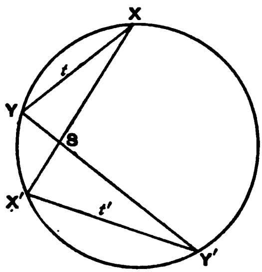
17. *O is a fixed point, and Q is a variable point on a fixed circle. QR is drawn such that the angle OQR is constant. Find the envelope of QR.*

Let QR be called p. Reciprocate for O. Then we have to find the locus of a point P taken on a tangent q to a conic one of whose foci is O, given that the angle between OP and q is constant. Draw OY the perpendicular from O on q. Then since the locus of Y is a circle and since $OY:OP$ is constant and $\angle YOP$ is constant, hence the locus of P is a circle; viz. the locus of Y enlarged in the ratio $OP:OY$ and turned through the angle YOP. Hence the envelope of p is a conic with O as one focus.

Ex. *If the locus of Q be a line instead of a circle, find the envelope of QR.*

18. To investigate bifocal properties of a conic by reciprocation we reflect the figure in the centre of the conic. For example—





In any central conic the pair of tangents from a point make equal angles with the focal radii to the point.

Let the tangents in Fig. i from T to a conic touch in P and Q. We have to prove that $PTS = QTS'$. Reflect the

whole figure in the centre C. The tangents at P and Q with their reflexions form a parallelogram $RTR'T'$. Then T' is the reflexion of T, Q' of Q, $T'Q'$ of TQ, $T'S$ of TS'. Hence the angle QTS' is equal to its reflexion, the angle $Q'T'S$. Hence we have to prove that $\angle STP$ and $\angle ST'Q'$ are equal. Reciprocating for S we obtain Fig. ii. Now

$$\angle STP = \angle ST,\ y = \angle SY,\ t = \angle SYX = \angle SX'Y'$$
$$= \angle SX',\ t' = \angle ST',\ x' = \angle ST'Q'.$$

Prove by reciprocation that—

Ex. 1. *The focal radii to a point on a conic make equal angles with the tangent at the point.*

Ex. 2. *The product of the perpendiculars from the foci of a conic on any tangent is equal to the square of the semi-axis minor.*

For in Fig. ii, $SX \cdot SX'$ is constant.

Ex. 3. *The sum of the reciprocals of the perpendiculars from any point O within a circle to the tangents from any point on the polar of O is constant.*

19. *To reciprocate a system of coaxal circles into a system of confocal conics.*

If we reciprocate the system of coaxal circles for any point O, we get a system of conics having one focus O in common. In order that the other focus may be common to all, the conics must have the same centre, i.e. the line at infinity must have the same pole for each conic. Hence in the figure of the circles, O must have the same polar for each circle, i.e. O must be one of the limiting points of the coaxal system. Now reciprocate the coaxal system for the limiting point L. Then the reciprocal conics have a focus and centre in common, and hence are confocal.

20. *To reciprocate a system of confocal conics into a system of coaxal circles.*

Since each conic is to be reciprocated into a circle, we must reciprocate for one of the common foci. Reciprocate for the focus O. Then since the conics have the same centre, the reciprocal circles have the same polar of O. We have to show that a system of circles each of which has the same polar

of O is coaxal. Drop the perpendicular OO' on the polar of O. Bisect OO' in X. Let OO' cut one of the circles in A, A'. Then since (OO', AA') is harmonic, and X bisects OO', hence $XA \cdot XA' = XO^2$, a constant. Hence X has the same power for all the circles. And the centres all lie on the line OO'. Hence the circles are coaxal, X being the foot of the radical axis.

Note that O, O' are the limiting points of the coaxal system.

The reciprocal of the other focus S is the radical axis.

For $OS = 2 \cdot OC$; hence $OS_1 = \frac{1}{2} \cdot OC_1$. But C_1 is the O' of the above proof. Hence S_1 is the X of the above proof.

Ex. 1. *The reciprocal of the minor axis is the other limiting point.*

Ex. 2. *S and H are the foci of a system of confocal conics. A parabola with S as focus touches the minor axis. Show that its directrix passes through H; and that if P, Q be the points of contact of a tangent to one of the confocals and the parabola, then PSQ is a right angle.*

We get a circle through the limiting points.

Ex. 3. *Deduce a property of coaxal circles from—'Tangents from any point to two confocals are equally inclined to each other.'*

Ex. 4. *Deduce a property of confocal conics from—'The polars of a fixed point for a system of coaxal circles meet at another fixed point; and the two points subtend a right angle at either limiting point.'*

Ex. 5. *If the sides of a variable polygon touch a conic, and all but one of the vertices lie on fixed confocal conics, the last vertex also lies on a fixed confocal conic.*

Reciprocate Poncelet's theorem respecting coaxal circles.

Reciprocation for any conic.

21. Having discussed the particular case of two reciprocal conics, one of which is a circle, we return to the general case of the reciprocal of a conic, taking any base conic.

The reciprocal of a conic, taking a conic with centre O as base conic, is a hyperbola, parabola, or ellipse, according as O is outside, on or inside the given conic.

Let α be the given conic and Γ the base conic, and α' the

reciprocal conic. Then a' is a hyperbola, parabola, or ellipse, according as the line at infinity cuts a' in real, coincident, or imaginary points. Now the reciprocal of the line at infinity is the pole of the line at infinity for Γ, i.e. is O. Hence the reciprocals of the points in which a' meets the line at infinity are the tangents to a from O. And the tangents from O are real if O be outside, coincident if O be on, and imaginary if O be inside a.

The reciprocal of *the centre* of the given conic, i.e. of the pole of the line at infinity for a, is the polar of O for a'. The reciprocal of the *asymptotes* of the given conic, i.e. of the tangents to a from the pole of the line at infinity for a, are the points of meet with a' of the polar of O for a', i.e. are the points of contact of tangents from O to a'.

As a particular case we may take the base conic to be a circle. Then the properties of § 8 and § 9 are also true.

Ex. 1. *The axes of the reciprocal of a conic for a point O are parallel to the bisectors of the angles between the tangents from O to the conic.*

Ex. 2. *The reciprocal of a parabola for any point on the directrix is a rectangular hyperbola.*

For since the points of contact of tangents from O to a subtend a right angle at O, hence the asymptotes of a' are perpendicular.

Ex. 3. Steiner's theorem. *The orthocentre of a triangle circumscribing a parabola is on the directrix.*

Let O be the orthocentre. Then we want to prove that the tangents from O are orthogonal; i.e. that the asymptotes of the reciprocal of the parabola for O are orthogonal; i.e. that the reciprocal is a r. h. But the reciprocal of a parabola passes through O. Hence the reciprocal passes through the vertices and the orthocentre of a triangle and is therefore a r. h.; for O is still the orthocentre.

Ex. 4. *The reciprocal of a rectangular hyperbola for any point O is a conic whose director passes through O.*

Ex. 5. *Reciprocate for any point—'A diameter of a rectangular hyperbola and the tangent at either end are equally inclined to either asymptote.'*

Let r be CP the diameter, q the tangent at P, and y the asymptote. Then we have to reciprocate that $\angle ry = \angle qy$.

We get—'If c be the polar of any point O on the director of a conic, and if from the point R on c a tangent be drawn touching in Q; then Y being either of the points in which c cuts the conic, RY and QY subtend equal angles at O.'

Ex. 6. *Show that two lines can be found such that the pairs of conjugate points on each for a given conic subtend right angles at a given point O.*

Ex. 7. *If the chord PQ of a conic subtend a right angle at a fixed point O on the conic, then PQ passes through a fixed point on the normal at O* (called the Frégier point of O for the conic).

Reciprocate for the fixed point; and we have to prove that the locus of the meet of perpendicular tangents of a parabola is a line (the directrix). On the normal as we see by taking P at O; then PQ becomes the normal.

Ex. 8. *Obtain by reciprocating Ex. 7 a property of a circle.*

Ex. 9. *Find the envelope of the chord of a conic which subtends a given angle at a given point on the conic.*

Ex. 10. *The envelope of a chord of a conic which subtends a right angle at a fixed point O, not on the conic, is a conic having a focus at O.*

Ex. 11. *A system of four-point conics or four-tangent conics can be reciprocated into concentric conics.*

Take as origin one of the vertices of the common self-conjugate triangle.

Ex. 12. *The reciprocal of a central conic, taking a concentric circle as base conic, is a similar conic.*

For $OA \,.\, OA_1 = OB \,.\, OB_1 = k^2$; hence

$$OA_1 : OB_1 :: OB : OA.$$

Ex. 13. *Reciprocate for any point—a system of coaxal circles.*

That is, a system of circles passing through the same two points, real or imaginary.

Ex. 14. *Reciprocate for any point O—'The directors of a system of conics touching the same four lines are coaxal.'*

Ex. 15. *Also—'The locus of the centres of a system of rectangular hyperbolas passing through the same three points is a circle.'*

***22.** *Reciprocate Carnot's theorem, taking any circle as base conic.*

Let O be the origin of reciprocation. Then, as in VI. 1, Carnot's theorem gives

$$\sin AOC_1 \,.\, \sin AOC_2 \ldots = \sin AOB_1 \,.\, \sin AOB_2 \ldots$$

Now $\angle AOC_1 = \angle ac_1$, and so on. Hence the reciprocal theorem is—'The sides a, b, c of a triangle meet in the points P, Q, R; and from P, Q, R are drawn the pairs of tangents $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ to any conic; then

$$\sin ac_1 . \sin ac_2 . \sin ba_1 . \sin ba_2 . \sin cb_1 . \sin cb_2$$
$$= \sin ab_1 . \sin ab_2 . \sin bc_1 . \sin bc_2 . \sin ca_1 . \sin ca_2,$$

where ac_1 denotes the angle between the lines a and c_1, and so on. And conversely if this relation hold, then the six lines $a_1 a_2 b_1 b_2 c_1 c_2$ touch the same conic.'

Ex. 1. *If the sides of a triangle ABC meet a conic in A_1A_2, B_1B_2, C_1C_2, then the six lines AA_1, AA_2, BB_1, BB_2, CC_1, CC_2 touch a conic; and conversely, if the latter touch a conic, the former are on a conic.*

Ex. 2. *Reciprocate the theorem—'The lines joining the vertices of a triangle to any two points meet the opposite sides in six points which lie on a conic.'*

VIII

1. Four conics c_1, c_2, c_3, c_4 have one focus and one tangent t in common. A second common tangent to c_1 and c_4 meets the corresponding directrix of c_1 at a point on t; similarly for c_2, c_4 and c_3, c_4. Show that the other common tangents of $c_1 c_2$, $c_2 c_3$, $c_3 c_1$ are concurrent.

2. ACB is the diameter of a circle whose centre is C. Two equal parabolas are drawn with foci at C and vertices at A and B. A hyperbola is drawn having a focus at C, and a vertex at D, one of the ends of the diameter perpendicular to AB, and touching the parabolas. The corresponding directrix of this hyperbola meets DC at E, and the hyperbola meets DC again at F. Show that

$$CF = 2\,CE = 3\,CD.$$

3. Reciprocate for the orthocentre of ABC the theorem—'If DEF be the feet of the perpendiculars from A, B, C on BC, CA, AB, then the radius of the circle about ABC is double the radius of the circle about DEF.'

4. Two conics having a common focus S touch at P. From any point Q on one of the conics, tangents are drawn to the other, meeting the tangent at P at U, V. The tangent at Q meets the tangent at P at T. Show that TU and TV subtend equal angles at S.

5. The common tangent of an ellipse and its circle of

curvature at P meets the tangent at P in a point T, such that SP and ST are equally inclined to the join of the focus S to the centre of curvature.

6. 'If two circles touch one another at C and be touched by a common tangent at A and B, then ACB is a right angle.' Reciprocate this theorem (i) for any point, (ii) for A, (iii) for C, and (iv) for the centre of one of the circles.

7. If two opposite vertices of a parallelogram circumscribed to a conic move on the directrices, the other two vertices move on the auxiliary circle.

8. The reciprocal for O of the focus of a parabola is the polar of the Frégier point of O for the reciprocal conic.

9. O, D, E are fixed points on a conic, and P a variable point. PD, PE meet the polar of the point in which chords which subtend a right angle at O meet, at B and C. Show that $\angle BOC = \angle DOE$.

10. Prove that if A, B the points of contact of a real common tangent of two conics which have a common focus subtend a right angle at that focus, the remaining tangents to the conics from A and B will intersect on the other real common tangent.

11. A perpendicular is drawn to any tangent of a parabola at the point where it meets a fixed tangent. Prove that the envelope of this line is a parabola whose axis is perpendicular to that of the given parabola and which touches the fixed tangent at the point where it is met by the directrix of the given parabola.

12. A, B are the common points of a system of coaxal circles and a fixed line through A meets any circle of the system at P. Show that the tangent at P always touches a fixed parabola whose focus is B.

13. PSQ is a focal chord of a conic, S being the focus. Prove that the perpendicular from S to the tangent at P will meet the tangent at Q also on the auxiliary circle.

14. Reciprocate with respect to one of the limiting points—'The orthogonal trajectory of a system of coaxal circles is another system of coaxal circles passing through the limiting points of the first system.'

15. P and U are fixed points on a conic. Through U are drawn two lines meeting the conic at L and M and the polar of the Frégier point of P at X and Y. Show that LM and XY subtend equal angles at P.

CHAPTER IX

ANHARMONIC OR CROSS RATIO

1. ONE of the *anharmonic* or *cross ratios* of the four collinear points A, B, C, D is $\dfrac{AB}{BC} \div \dfrac{AD}{DC}$. This is denoted by (AC, BD). So every other order of writing the letters gives us a cross ratio of the points, e.g. another cross ratio is

$$(BA, CD) = \frac{BC}{CA} \div \frac{BD}{DA}.$$

Ex. 1. *If* $(AB, CD) = (AB, C'D')$, *then* $(AB, CC') = (AB, DD')$.

Ex. 2. *If* $(AC, A'B) = (A'C', AB')$, *then* $(AC, C'B) = (A'C', CB')$.

Ex. 3. *If* $(AB, CD) = (A'B', C'D')$, *and* $(AB, CE) = (A'B', C'E')$, *show that* $(AB, DE) = (A'B', D'E')$.

Ex. 4. *If* OA, OB, OC *cut* BC, CA, AB *in* P, Q, R, *and if any line cut* BC, CA, AB *in* P', Q', R', *then*

$$(BC, PP') \times (CA, QQ') \times (AB, RR') = -1.$$

Ex. 5. *A cross ratio is not altered by inversion for a point on the line.*

For given $OA \,.\, OA' = OB \,.\, OB' = \ldots = k^2$,

we have $AB = OB - OA = k^2/OB' - k^2/OA'$
$= -k^2 \,.\, A'B'/OA' \,.\, OB'$.

2. *A cross ratio is equal to any other in which, any two points being interchanged, the other two are also interchanged.*

Let (AC, BD) be the cross ratio. We may interchange A with B, C or D. Hence we have to prove that

$$(AC, BD) = (BD, AC) = (CA, DB) = (DB, CA),$$

or that

$$\frac{AB}{BC} \cdot \frac{DC}{AD} = \frac{BA}{AD} \cdot \frac{CD}{BC} = \frac{CD}{DA} \cdot \frac{BA}{CB} = \frac{DC}{CB} \cdot \frac{AB}{DA}.$$

3. *There are 24 cross ratios of four points; and these can be divided into 3 groups of 8, such that every cross ratio in a group is equal to or the reciprocal of every other in the group.*

Let the points be $ABCD$. Take the three cross ratios (AB, CD), (AC, DB) and (AD, BC), got by changing BCD cyclically to CDB, DBC. Now

$$(AB, CD) = (BA, DC) = (CD, AB) = (DC, BA)$$

by IX. 2. Also it is easy to prove that (AB, CD) is the reciprocal of (AB, DC), (BA, CD), (CD, BA), (DC, AB). Hence we get a group of 8 connected with (AB, CD). Similarly there is a group of 8 connected with (AC, DB) and with (AD, BC). And no ratio can belong to two groups; for in the first group AB are together and CD, so in the second group AC are together and DB, and in the third group AD and BC.

4. *If* $\lambda = (AB, CD)$, $\mu = (AC, DB)$, $\nu = (AD, BC)$,

then $$\lambda + \frac{1}{\mu} = \mu + \frac{1}{\nu} = \nu + \frac{1}{\lambda} = -\lambda\mu\nu = 1.$$

For $$\lambda + \frac{1}{\mu} - 1 = \frac{AC}{CB} \cdot \frac{DB}{AD} + \frac{DC}{AD} \cdot \frac{AB}{BC} - 1$$

$$= \frac{AC.DB - DC.AB - CB.AD}{CB.AD}$$

$$= \frac{c(b-d) - (c-d)b - (b-c)d}{CB.AD}$$

taking A as origin

$$= \frac{cb - cd - cb + db - bd + cd}{CB.AD} = 0.$$

Also $$\lambda.\mu.\nu = \frac{AC}{CB} \cdot \frac{DB}{AD} \cdot \frac{AD}{DC} \cdot \frac{BC}{AB} \cdot \frac{AB}{BD} \cdot \frac{CD}{AC} = -1.$$

We have now shown that the three fundamental cross ratios λ, μ, ν are connected by the above four relations. Two of these are independent and give μ, ν in terms of λ. The other two can be derived from these. Hence given any one cross ratio of four points, the other 23 can be calculated.

Ex. 1. *Show that no real range can be found in which*

$$\lambda = \mu = \nu.$$

For $\lambda^2 - \lambda + 1 = 0$ if $\lambda = \mu = \nu$.

Ex. 2. *Of the three λ, μ, ν, two are positive and one negative.*

Ex. 3. *If any cross ratio of the range $ABCD$ is equal to the corresponding cross ratio of the range $A'B'C'D'$, then every two corresponding cross ratios of the ranges are equal.*

For if $\lambda = \lambda'$, then $\mu = \mu'$ since $\lambda + \dfrac{1}{\mu} = 1$; so $\nu = \nu'$.

Two such ranges are said to be *homographic*, and we denote the fact by the equation $(ABCD) = (A'B'C'D')$.

Ex. 4. *If* $(ABB'C) = (A'B'BC')$
and $(ABB'D) = (A'B'BD')$,
show that $(BB'CD) = (B'BC'D')$.

Divide

$(BB', AC) = (B'B, A'C')$ by $(BB', AD) = (B'B, A'D')$.

Ex. 5. *If $(ABCD) = (A'B'C'D')$, prove that*

$$\frac{AB.CD}{A'B'} + \frac{AC.DB}{A'C'} + \frac{AD.BC}{A'D'} = 0.$$

We have to show that

$$\frac{AB.CD}{AD.BC}\cdot\frac{1}{A'B'} + \frac{AC.DB}{AD.BC}\cdot\frac{1}{A'C'} + \frac{1}{A'D'} = 0.$$

But $\dfrac{AB.CD}{AD.BC} = -\dfrac{BA}{AD} \div \dfrac{BC}{CD} = -(BD, AC)$

$$= -(B'D', A'C') = \frac{A'B'.C'D'}{A'D'.B'C'}, \text{ and so on.}$$

Hence the relation reduces to

$$C'D' + D'B' + B'C' = 0.$$

Ex. 6. *If $(ABCD) = (A'B'C'D')$ and O' be any point on $A'B'$, prove that*

$$\frac{AB.CD}{A'B'}.O'B' + \frac{AC.DB}{A'C'}.O'C' + \frac{AD.BC}{A'D'}.O'D' = 0.$$

5. *If two points of a range of four points coincide, each of the cross ratios is equal to 0, 1 or ∞; and no cross ratio of four points can equal 0 or 1 or ∞ unless two points coincide.*

Call the two coincident points A and B. Then

$$\lambda = \frac{AC}{CB}\cdot\frac{DB}{AD} = -1 \times -1 = 1$$

since $AC : CB = -CA : CB = -1$, so $DB : AD = -1$;

$$\mu = \frac{AD}{DC}\cdot\frac{BC}{AB} = \infty \text{ since } AB = 0;$$

$$\nu = \frac{AB}{BD}\cdot\frac{CD}{AC} = 0 \text{ since } AB = 0.$$

Hence $\lambda^{-1} = 1$, $\mu^{-1} = 0$, $\nu^{-1} = \infty$.

Conversely, if $\frac{AC}{CB}\cdot\frac{DB}{AD} = 0$, $AC = 0$ or $DB = 0$, i.e. A and C coincide or B and D coincide; if

$$\frac{AC}{CB}\cdot\frac{DB}{AD} = 1,\quad \frac{AC}{CB} = \frac{AD}{DB},$$

hence unless A and B coincide, C and D must coincide, since they divide AB in the same ratio; if

$$\frac{AC}{CB}\cdot\frac{DB}{AD} = \infty,\quad CB = 0 \text{ or } AD = 0,$$

hence B and C coincide or A and D coincide.

If (AC, BD) be harmonic, then $(AC, BD) = -1$.

For $\frac{AB}{BC} = -\frac{AD}{DC}$, hence $\frac{AB}{BC} \div \frac{AD}{DC} = -1$.

If (AC, BD) be harmonic, then $(AC, BD) = (AC, DB)$; and conversely, if $(AC, BD) = (AC, DB)$, then either (AC, BD) is harmonic or two points coincide.

For $(AC, BD) = (AC, DB)$,

if $\frac{AB}{BC}\cdot\frac{DC}{AD} = \frac{AD}{DC}\cdot\frac{BC}{AB}$, i.e. if $\left(\frac{AB}{BC}\cdot\frac{DC}{AD}\right)^2 = 1$,

i.e. if $\frac{AB}{BC}\cdot\frac{DC}{AD} = \pm 1$, i.e. if $(AC, BD) = \pm 1$.

If $(AC, BD) = +1$, then A and C, or B and D coincide; and if $(AC, BD) = -1$, then (AC, BD) is harmonic; and conversely.

If a range of four points is harmonic, each of the cross ratios is equal to -1, $\frac{1}{2}$, or 2; and, conversely, if a cross ratio of four points is equal to -1 or $\frac{1}{2}$ or 2, the range is harmonic.

If (AC, BD) is harmonic, then $\frac{AB}{BC} \cdot \frac{DC}{AD} = -1$. Hence $\mu = \frac{AD}{DC} \cdot \frac{BC}{AB} = -1$; hence $\lambda + \frac{1}{\mu} = 1$ gives $\lambda = 2$ and $\nu + \frac{1}{\lambda} = 1$ gives $\nu = \frac{1}{2}$.

Conversely if $(AC, BD) = -1$, then $\frac{AB}{BC} \cdot \frac{DC}{AD} = -1$, hence (AC, BD) is harmonic; if $(AC, BD) = \frac{1}{2}$, $\mu = (AC, BD)^{-1} = 2$, $\therefore\ \mu + \frac{1}{\nu} = 1$ gives $\nu = -1$, hence (AD, BC) is harmonic; if $(AC, BD) = 2$, $\mu = \frac{1}{2}$, $\therefore\ \lambda + \frac{1}{\mu} = 1$ gives $\lambda = -1$, hence (AB, CD) is harmonic.

A convenient abbreviation of the statement that (AB, CD) is harmonic is that $(AB, CD) = -1$.

6. *If A, B, C, D, D' be collinear points, such that*

$$(AC, BD) = (AC, BD'),$$

then D and D' coincide,

For $\frac{AB}{BC} \cdot \frac{DC}{AD} = \frac{AB}{BC} \cdot \frac{D'C}{AD'}$, hence $\frac{DC}{AD} = \frac{D'C}{AD'}$;
i.e. AC is divided in the same ratio at D and D'; hence D and D' coincide.

7. *If four lines a, b, c, d passing through the same point V be cut by two transversals in $ABCD$ and $A'B'C'D'$,* then

$$(ABCD) = (A'B'C'D').$$

It is sufficient to prove that

$$(AC, BD) = (A'C', B'D').$$

Now
$$(AC, BD) = \frac{AB}{BC} \cdot \frac{DC}{AD} = \frac{\Delta AVB}{\Delta BVC} \cdot \frac{\Delta DVC}{\Delta AVD}$$
$$= \frac{VA\,.\,VB\,.\sin AVB}{VB\,.\,VC\,.\sin BVC} \cdot \frac{VD\,.\,VC\,.\sin DVC}{VA\,.\,VD\,.\sin AVD}$$
$$= \frac{\sin AVB}{\sin BVC} \cdot \frac{\sin DVC}{\sin AVD}.$$

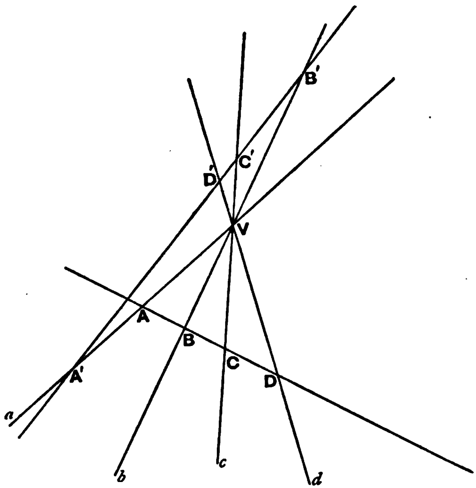
Similarly,

$$(A'C', B'D') = \frac{\sin A'VB'}{\sin B'VC'} \cdot \frac{\sin D'VC'}{\sin A'VD'}.$$

Now AVB is equal to either $A'VB'$ or its supplement. In either case, $\sin AVB = \sin A'VB'$. And so on. Hence

$$(AC, BD) = (A'C', B'D'), \text{ i.e. } (ABCD) = (A'B'C'D').$$

We may enunciate the above theorem in the form—*Every transversal cuts a pencil of four lines in the same cross ratio.*



The cross ratio (AC, BD) of the pencil is written

$$V(AC, BD) \text{ or } (ac, bd).$$

Also, by the above,

$$(ac, bd) = \frac{\sin ab}{\sin bc} \div \frac{\sin ad}{\sin dc}.$$

If we draw the section $A'B'C'D'$ parallel to VD, then D' is at infinity and

$$(AC, BD) = (A'C', B'D') = \frac{A'B'}{B'C'} \div \frac{A'D'}{D'C'} = -\frac{A'B'}{B'C'},$$

since $A'D' : C'D'$ is equal to 1 when D' is at infinity. Hence every cross ratio can be expressed as a simple ratio by projecting one of the points to infinity.

Ex. 1. *Show that the fundamental cross ratios* λ, μ, ν *of the range* $(ABCD)$ *are equal to* $\operatorname{cosec}^2\phi$, $-\tan^2\phi$ *and* $\cos^2\phi$, *where* 2ϕ *is the angle at which the circles on* AC *and* BD *as diameters intersect.*

For if the circles intersect at P

$$\lambda = \frac{AC}{CB}\cdot\frac{DB}{AD} = \frac{\sin APC}{\sin CPB}\cdot\frac{\sin DPB}{\sin APD},$$

also $\angle APC = \frac{\pi}{2}$, $\angle DPB = -\frac{\pi}{2}$; and it can be proved that

$$\angle CPB = -\phi,\quad \angle APD = \pi - \phi.$$

Ex. 2. *Given the three points* A, B, C; *find* D *so that* (AB, CD) *may have a given value* λ.

Take any line AB', and divide it in C' so that

$$-AC' \div C'B' = \lambda.$$

Let BB', CC' meet in V. Through V draw VD parallel to AB'. Then $(AB, CD) = (AB', C'\Omega')$, [where Ω' is the point at infinity upon AB',] $= -AC' \div C'B' = \lambda$.

Ex. 3. *Through a given point* O *draw a transversal to cut the sides of a given triangle* ABC *in points* A', B', C', *such that* $(OA', B'C')$ *may have a given value.*

Let OA cut BC in O'. Then

$$(OA', B'C') = A(OA', B'C') = (O'A', CB).$$

Hence A' is known.

Ex. 4. *If the ranges* (OA', BC), (OB', CA), (OC', AB) *are harmonic, show that* $(OABC) = (OA'B'C')$.

Project O to infinity. Then A', B', C' bisect BC, CA, AB; and we have to show that $AB : BC = A'B' : B'C'$. Now take A as origin.

Ex. 5. *Prove the relation* $\lambda + \dfrac{1}{\mu} = 1$ *by projection.*

Projecting D to infinity, we get

$$\lambda = -\frac{A'C'}{C'B'},\quad \mu = -\frac{B'C'}{A'B'}.$$

Hence $$\lambda + \frac{1}{\mu} = -\frac{A'C'}{C'B'} + \frac{A'B'}{C'B'} = \frac{C'B'}{C'B'} = 1.$$

Ex. 6. *If one cross ratio of the pencil* $V(ABCD)$ *is equal to the corresponding cross ratio of the pencil* $V'(A'B'C'D')$, *then every cross ratio is equal to the corresponding cross ratio.*

Consider sections of the pencils and use Ex. 3 of § 4.

Two such pencils are said to be *homographic*, and we denote the fact by the equation $V(ABCD) = V'(A'B'C'D')$.

Ex. 7. *If* $V(ABCD) = V'(A'B'C'D')$, *then*

$$\frac{\sin AVB \,.\, \sin CVD}{A'B'} + \frac{\sin AVC \,.\, \sin DVB}{A'C'} + \frac{\sin AVD \,.\, \sin BVC}{A'D'} = 0.$$

8. *A cross ratio of a range of four points is unaltered by projection.*

Let the range $ABCD$ be joined to the vertex V, and let the joining plane cut the plane of projection in $A'B'C'D'$. Then since $A'B'C'D'$ is a section of the pencil $V(ABCD)$, it follows that $(ABCD) = (A'B'C'D')$.

9. *A cross ratio of a pencil of four lines is unaltered by projection.*

Join the pencil $O(ABCD)$ to the vertex V, and let the joining planes cut the plane of projection in the pencil $O'(A'B'C'D')$. Through V draw any plane cutting the pencils in $abcd$ and $a'b'c'd'$. Then

$$O(ABCD) = (abcd) = V(abcd) = V(a'b'c'd')$$
$$= (a'b'c'd') = O'(A'B'C'D').$$

Hence the pencils $O(ABCD)$ and $O'(A'B'C'D')$ have the same cross ratios.

Ex. *The figure ABCD consisting of four points joined by four lines can be projected into any figure $A'B'C'D'$ of the same kind.*

Let AC, BD meet in U, and $A'C'$, $B'D'$ meet in U'. Take X on AC so that $(XAUC) = (\Omega' A'U'C')$, and Y on BD so that $(YBUD) = (\Omega B'U'D')$, where Ω and Ω' are at infinity. Now project XY to infinity, and the angles AUB, BAU into angles of magnitude $A'U'B'$, $B'A'U'$. Let the projections of $ABCDUXY$ be $a'b'c'd'u'\omega'\omega$, where ω and ω' are at infinity. Then $(\omega' a'u'c') = (XAUC) = (\Omega' A'U'C')$. Hence

$$a'u' : u'c' :: A'U' : U'C';$$

so $b'u' : u'd' :: B'U' : U'D'$; also $\angle a'u'b' = \angle A'U'B'$ and

$$\angle b'a'u' = \angle B'A'U'.$$

Hence the figures $a'b'c'd'u'$ and $A'B'C'D'U'$ are similar. If they are not equal, we proceed as in IV. 7.

Note that this construction fails if XY as constructed be at infinity; in other cases, by IV. 6, the construction is real.

Cross ratio of four planes meeting in a line.

10. *Any transversal cuts four planes which pass through the same line in four points whose cross ratio is constant.*

Let two transversals cut the planes in $ABCD$ and $A'B'C'D'$. Join $ABCD$ to any point O on the meet of the planes, and $A'B'C'D'$ to any other point O' on this meet. Then the meet of the planes $OABCD$ and $O'A'B'C'D'$ is a line which cuts the four given planes in the points $\alpha, \beta, \gamma, \delta$, say.

Then $(ABCD) = O(ABCD) = O(\alpha\beta\gamma\delta) = (\alpha\beta\gamma\delta)$
$= O'(\alpha\beta\gamma\delta) = O'(A'B'C'D') = (A'B'C'D')$.

Hence $(ABCD)$ is constant.

Any plane cuts four planes which meet in a line in four lines whose cross ratio is constant.

Let any two planes cut the intersections of the four planes at O and O'; and let the intersection of the two planes cut the four planes at $\alpha, \beta, \gamma, \delta$.

Then $O(\alpha\beta\gamma\delta) = (\alpha\beta\gamma\delta) = O'(\alpha\beta\gamma\delta)$.

Homographic ranges and pencils.

11. Two *ranges* of points $ABCD\ldots$ and $A'B'C'D'\ldots$ on the same or different lines, in which to each point (A say) of one range corresponds a point (A') of the other, are said to be *homographic* if every cross ratio of every four points $(ABCD)$ of one range is equal to the corresponding cross ratio of the corresponding four points $(A'B'C'D')$ of the other.

So the *pencils* $V(ABC\ldots)$ and $V'(A'B'C'\ldots)$ are said to be *homographic* if every cross ratio such as $V(AB, CD)$ is equal to the corresponding cross ratio $V'(A'B', C'D')$. Also a *range* $(ABC\ldots)$ and a *pencil* $V'(A'B'C'\ldots)$ are said to be *homographic* if every cross ratio such as (AB, CD) is equal to the corresponding cross ratio $V'(A'B', C'D')$.

It is convenient to abbreviate the statement that the ranges $(ABC\ldots)$ and $(A'B'C'\ldots)$ are homographic into

$$(ABC\ldots) = (A'B'C'\ldots).$$

So $V(ABC...) = V'(A'B'C'...)$ means that the pencils $V(ABC...)$ and $V(A'B'C'...)$ are homographic; and

$$(ABC...) = V'(A'B'C'...)$$

means that the range $(ABC...)$ and the pencil $V'(A'B'C'...)$ are homographic.

12. *Two ranges which are homographic with a third range are homographic with one another.*

For if the homographic ranges $(ABC...)$ and $(A'B'C'...)$ are homographic with $(A''B''C''...)$, then, taking any cross ratio (AB, CD), we have $(AB, CD) = (A''B'', C''D'')$ and $(A'B', C'D') = (A''B'', C''D'')$. Hence

$$(AB, CD) = (A'B', C'D')$$

is true for every cross ratio. Hence the ranges $(ABC...)$ and $(A'B'C'...)$ are homographic.

In exactly the same way we prove that (writing f for either a range or a pencil) if f_1 and f_2 are both homographic with f_3, then f_1 and f_2 are homographic with one another. This we abbreviate into $f_1 = f_3 = f_2$, $\therefore$ $f_1 = f_2$. So if f_1 is homographic with f_2 and f_2 with f_3, ... and f_{n-1} with f_n, we conclude that f_1 is homographic with f_n, or briefly, if

$$f_1 = f_2 = f_3 = ... = f_{n-1} = f_n, \text{ then } f_1 = f_n.$$

13. *Homographic ranges exist;* for if we join the points of the range $(ABC...)$ to any point V and take any section $(A'B'C'...)$ of the pencil $V(ABC...)$, then

$$(A'B'C'...) = (ABC..\,).$$

For let (AB, CD) be any cross ratio of the range $(ABC...)$. Then $(A'B', C'D') = (AB, CD)$ by § 7.

Homographic pencils exist; for if $(ABC...)$ is any range and U and V any two points, the pencils $U(ABC...)$ and $V(ABC...)$ are homographic. For

$$U(AB, CD) = (AB, CD) = V(AB, CD).$$

A pencil homographic with a range can be found; for as proved above $V(ABC...)$ and its section $(ABC...)$ are homographic.

Notice that we have incidentally proved that—

Two sections of the same pencil are homographic. Pencils

standing on the same range are homographic. A pencil is homographic with the range which it determines on any transversal.

14. *Two ranges* $ABC\ldots$ and $A'B'C'\ldots$ on different lines are said to be *in perspective* when the lines AA', BB', CC', ... joining corresponding points meet in a point (called the *centre of perspective*).

Two pencils $V(ABC\ldots)$ and $V'(A'B'C'\ldots)$ at different vertices are said to be *in perspective* when the meets of corresponding rays lie on a line (called the *axis of perspective*.)

Two ranges in perspective are homographic.

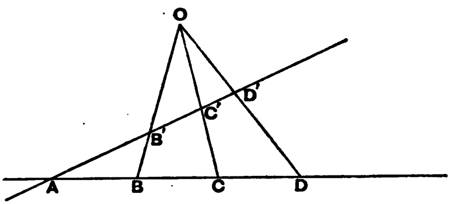
For they are sections of the same pencil.

Two pencils in perspective are homographic.

For they stand on the same range.

15. *If two homographic ranges on different lines have the meet of the lines as a point corresponding to itself in the two ranges, then the ranges are in perspective.*

Let the ranges be $(ABCD\ldots) = (AB'C'D'\ldots)$.



Let BB', CC' meet in O, and let OD meet AB' in D''. Then $(AB'C'D') = (ABCD)$ by hypothesis $= (AB'C'D'')$ by projection. Hence $(AB'C'D') = (AB'C'D'')$, i.e. D' and D'' coincide, i.e. the join DD' of any pair of corresponding points passes through O.

Any two homographic ranges can be placed so as to be in perspective; viz. by placing any point A of one range on the corresponding point A' of the other range.

Three pairs of corresponding points of two homographic ranges completely determine the ranges.

For on placing A' on A, BB', CC' determine O and then OD cuts AB' at D'.

If $(AB, CP) = (A'B', C'P')$ *where* P *and* P' *are variable points on the lines* ABC *and* $A'B'C'$, *then* P *and* P' *generate homographic ranges of which* A *and* A', B *and* B', C *and* C' *are corresponding points.*

For place A' on A. Then AA', BB', CC', PP' meet in a point. Hence the ranges $(ABC...P...)$ and $(A'B'C'...P'...)$ are in perspective and therefore homographic.

Ex. 1. *If* A *be the meet of two corresponding rays of two homographic pencils, then any line through* A *will cut one pencil in a range in perspective with the range determined on any other line through* A *by the other pencil.*

Ex. 2. AO *meets* BC *at* D *and* BO *meets* CA *at* E. *On* AD *and* BE *are taken* X *and* Y *such that* (AD, OX) *and* (BE, OY) *are harmonic. Show that* XY *passes through* C.

AE and DB pass through C. Hence XY will pass through C if $(OADX) = (OEBY)$. But

$$(OX, AD) = -1 = (OY, EB).$$

Ex. 3. *The points* A *and* B *move on fixed lines through* O, *and* U *and* V *are fixed points collinear with* O; *if* UA *and* VB *meet on a fixed line, show that* AB *passes through a fixed point.*

Take several positions of the point A, viz. $A_1 A_2 A_3 \ldots$. Join $A_1 U$ cutting the given line in C_1, and join $C_1 V$ cutting OB in B_1. Similarly construct $C_2 C_3 \ldots$ and $B_2 B_3 \ldots$. Then it is sufficient to prove that the ranges $A_1 A_2 \ldots$ and $B_1 B_2 \ldots$ are in perspective, i.e. that $(OA_1 A_2 \ldots) = (OB_1 B_2 \ldots)$. But $(OA_1 A_2 \ldots) = U(OA_1 A_2 \ldots) = (XC_1 C_2 \ldots)$ if UV and $C_1 C_2$ intersect at $X = V(XC_1 C_2 \ldots) = (OB_1 B_2 \ldots)$. Hence

$$(OA_1 A_2 \ldots) = (OB_1 B_2 \ldots).$$

Ex. 4. *If the points* A, B, C *move on fixed lines through* O, *and* AB *turn about a fixed point* P, *and* BC *turn about a fixed point* Q, *show that* CA *turns about a fixed point.*

Ex. 5. *If the given triangle* ABC *be circumscribed to the given triangle* LMN, *prove the following construction for inscribing a triangle* PQR *in the triangle* LMN *which shall also be circumscribed to the triangle* ABC, *viz.: take any point* R *on* LM, *let* AR *cut* NL *at* Q *and let* BR *cut* MN *at* P; *then will* PQ *pass through* C.

16. *If two homographic pencils at different vertices have the ray*

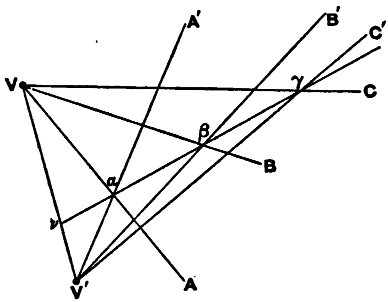
joining the vertices as a ray corresponding to itself in the two pencils, then the pencils are in perspective.

Let the two homographic pencils be $V(V'ABC\ldots)$ and $V'(VA'B'C'\ldots)$. Let VA cut $V'A'$ in a. Let VB cut $V'B'$ in β. Let $a\beta$ cut VV' in v. If $a\beta$ does not cut VC and $V'C'$ in the same point, let $a\beta$ cut VC in γ and $V'C'$ in γ'.

Now $V(V'ABC\ldots) = V'(VA'B'C'\ldots)$. Hence

$$(va\beta\gamma) = (va\beta\gamma'),$$

by considering the sections of these pencils by $a\beta$. Hence γ and γ' coincide. Hence VC, $V'C'$ meet on $a\beta$. So every



pair of corresponding rays meet on $a\beta$. Hence the pencils are in perspective.

Any two homographic pencils can be placed so as to be in perspective; viz. by placing any ray of one pencil so as to be in the same line as the corresponding ray of the other pencil.

Three pairs of corresponding rays of two homographic pencils completely determine the pencils.

For placing one ray on the corresponding ray, VA, VA' determine a and VB, VB' determine β. Then $V'C'$ is the line joining V' to the intersection of VC and $a\beta$.

If $V(AB, CP) = V'(A'B', C'P')$ *where* VP *and* $V'P'$ *are variable rays through* V *and* V', *then* VP *and* $V'P'$ *generate homographic pencils of which* VA *and* $V'A'$, VB *and* $V'B'$, VC *and* $V'C'$ *are corresponding rays.*

For if we place VA on $V'A'$, the points $(VB;\ V'B')$,

$(VC;\ V'C')$, and $(VP;\ V'P')$ lie on a line. Hence the pencils $V(ABC...P...)$ and $V'(A'B'C'...P'...)$ are in perspective and therefore homographic.

Ex. 1. *If $(ABCD...)$ and $(A'B'C'D'..)$ be two homographic ranges, and any two points V, V' be taken on AA', show that the meets of VB and $V'B'$, of VC and $V'C'$, of VD and $V'D'$, &c., all lie on a line.*

Ex. 2. *If AB pass through a fixed point U, and A and B move on fixed lines meeting in O, and if V, W be fixed points collinear with O, show that the locus of the meet of AV and BW is a line.*

Let AV and BW cut in P. Take several positions $A_1A_2...$ of A, $B_1B_2...$ of B, $P_1P_2...$ of P. Then it is sufficient to prove that $V(P_1P_2...)$ and $W(P_1P_2...)$ are in perspective, i.e. that $V(OP_1P_2...) = W(OP_1P_2...)$. But

$$V(OP_1P_2...) = (OA_1A_2...) = U(OA_1A_2...)$$
$$= (OB_1B_2...) = W(OP_1P_2...).$$

Ex. 3. *Show that the meet of UV and OB, and the meet of UW and OA lie on the locus.*

Let UV meet OA at A' and OB at B'. Then VA' and WB' meet at B' which is therefore a position of P. So for $(UW;\ OA)$.

Ex. 4. *If A and B move on fixed lines through O, and AB, BP, and AP pass through fixed collinear points U, V, W, show that the locus of P is a line through O.*

Ex. 5. *Show that VP and $V'P'$ generate homographic pencils if*

$$\frac{\sin AVP}{\sin BVP}\cdot\frac{\sin BVC}{\sin AVC}+\frac{\sin C'V'P'}{\sin B'V'P'}\cdot\frac{\sin B'V'A'}{\sin C'V'A'}=1.$$

Taking sections of the pencils, this is

$$(AB, PC)+(A'P', C'B')^{-1}=1.$$

But
$$(AB, PC)+(AP, CB)^{-1}=1,$$
$$\therefore\ (AP, CB)=(A'P', C'B').$$

Ex. 6. *If $\sin AVP/\sin BVP \div \sin A'V'P'/\sin B'V'P'$ is constant, show that VP and $V'P'$ generate homographic ranges.*

Let C and C' be positions of P and P'. Then the above gives $V(AB, CP) = V'(A'B', C'P')$.

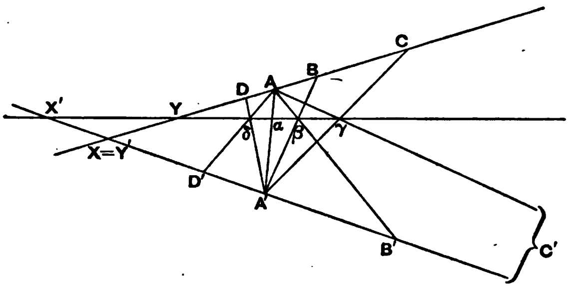
17. *If $(ABC...)$ and $(A'B'C'...)$ be two homographic ranges on different lines, then the meet of AB' and $A'B$, of BC' and $B'C$, and generally of PQ' and $P'Q$, where PP', QQ are any two*

pairs of corresponding points, all lie on a line (called *the homographic axis*).

Let the two lines meet in a point which we shall call X or Y', according as we consider it to belong to the range $(ABC...)$ or to $(A'B'C'...)$. Take the points X' and Y corresponding to the point $X (= Y')$ in the two ranges. Then every cross meet such as $(PQ'; P'Q)$ lies on $X'Y$. For by hypothesis $(XYABC...) = (X'Y'A'B'C'...)$. Hence

$$A'(XYABC...) = A(X'Y'A'B'C'...);$$

and these two pencils have the common ray AA'; hence



they are in perspective; hence $(A'X; AX')$, $(A'Y; AY')$, $(A'B; AB')$,... all lie on a line. But $(A'X; AX')$ is X', and $(A'Y; AY')$ is Y. Hence $(A'B; AB')$ lies on the fixed line $X'Y$; i. e. every cross meet lies on a fixed line, for AA', BB' are any two pairs of corresponding points.

18. By Reciprocation (see XIII. 1), or by a similar proof, we show that *if $V(ABCD...)$ and $V'(A'B'C'D'...)$ be homographic pencils, then all the cross joins, such as the join of $(VB; V'C')$ with $(V'B'; VC)$, pass through a fixed point* (called *the homographic pole*).

Ex. 1. *If A, B, C be any three points on a line, and A', B', C' be any three points on another line, show that the meets of AB' and $A'B$, of AC' and $A'C$, and of BC' and $B'C$, are collinear.*

Consider $X (= Y')$ as above.

Ex. 2. *When two ranges are in perspective, the axis of homography is the polar of the centre of perspective for the lines of the ranges.*

Projective ranges and pencils.

19. If *range* a is in perspective with range β, and range β with range γ, and range γ with range δ, and so on; then each of the ranges a, β, γ, δ ... is said to be *projective* with every other.

If *pencil* a is in perspective with pencil β, and pencil β with pencil γ, and pencil γ with pencil δ, and so on; then each of the pencils a, β, γ, δ ... is said to be *projective* with every other.

Projective ranges are homographic.

For the range a is homographic with the range β, being in perspective with it; so β with γ, γ with δ, and so on; hence each is homographic with every other.

Projective pencils are homographic.

For the pencil a is homographic with the pencil β, being in perspective with it; and so on.

Homographic ranges are projective.

For they can be put in perspective with the same range on the homographic axis.

Homographic pencils are projective.

For they can be put in perspective with the same pencil at the homographic pole.

A range and a pencil are said to be *projective*, when the range is projective with a section of the pencil.

Hence *a range and a pencil which are projective are homographic;* and *a range and a pencil which are homographic are projective.*

IX

1. If through the vertices $A, B, C, \ldots$ of a polygon there be drawn any lines $AA', BB', CC', \ldots$ then the continued product of the ratios $\sin ABB'/\sin B'BC$ is unaltered by projection.

2. A variable circle passes through a fixed point and cuts a given line at a given angle. Show that it determines on the line two homographic ranges.

3. Six points A, B, C, D, E, F are taken, such that AB, FC, ED are concurrent, and also FA, EB, DC. Show that BC, AD, FE are concurrent.

4. If the vertices of a polygon move on fixed concurrent lines, and all but one of the sides pass through fixed points, this side and every diagonal will pass through a fixed point.

5. If each side of a polygon pass through one of a set of collinear points whilst all but one of its vertices slide on fixed lines, then will the remaining vertex and every meet of two sides describe a line.

6. P, Q, R are any points on MN, NL, LM. A is taken on MN, AQ meets LM at B, BP meets NL at C, CR meets MN at A', $A'Q$ meets LM at B', $B'P$ meets NL at C'. Show that $C'A$ passes through R.

CHAPTER X

HOMOGRAPHIC RANGES AND PENCILS

1. THE points corresponding to the two points at infinity in two homographic ranges are called the *vanishing points.*

To construct the vanishing points.

Let the ranges be $(ABC\ldots)$ on the line l and $(A'B'C'\ldots)$ on the line l'. Place A' on A so that l' does not coincide with l. The ranges are now in perspective. Hence BB', CC', ... all pass through the same point O. Let Ω and Ω' be the points at infinity on l and l', and J' and I the points corresponding to Ω and Ω'; so that I and J' are the vanishing points on l and l'. Then since I and Ω' are corresponding points $I\Omega'$ passes through O; i.e. I is the intersection of l with the parallel through O to l'. So OJ' is parallel to l.

2. *In two homographic ranges* $(ABCP\ldots)$ *and* $(A'B'C'P'\ldots)$, *on the same or different lines, if I correspond to the point Ω' at infinity in the range $(A'B'\ldots)$, and J' correspond to the point Ω at infinity in the range $(AB\ldots)$, then $IP \,.\, J'P'$ is the same whatever corresponding points P and P' are taken.*

For we have $(I\Omega ABCP\ldots) = (\Omega' J' A'B'C'P'\ldots)$;

hence $(AP, I\Omega) = (A'P', \Omega' J')$,

i.e. $AI/IP \div A\Omega/\Omega P = A'\Omega'/\Omega' P' \div A'J'/J'P'$.

But $A\Omega/\Omega P = -1$ and $A'\Omega'/\Omega' P' = -1$.

$$\therefore \quad AI/IP = J'P'/A'J',$$

$$\therefore \quad IP \,.\, J'P' = IA \,.\, J'A', \quad \text{which is constant.}$$

Conversely, *if $IP \,.\, J'P'$ be constant, then P and P' generate ranges which are homographic, and I and J' are the points corresponding to the points at infinity in the ranges.*

For let A and A' be any positions of P and P', then $IP \,.\, J'P' = IA \,.\, J'A'$. Hence retracing the above steps, we

get $(AP,\ I\Omega) = (A'P',\ \Omega'J')$. Hence P and P' are corresponding points in the ranges determined by $AI\Omega$ and $A'\Omega'J'$, and I and J' correspond to Ω' and Ω in these ranges.

We can also give a geometrical proof: For, with the same figure as in § 1, we have $IP:IO::J'O:J'P'$; hence

$$IP \cdot J'P' = IO \cdot J'O = IA \cdot J'A' \text{ similarly.}$$

Ex. 1. *If $OP \cdot OP'$ be constant, O being the meet of the lines on which P and P' lie, show that P and P' generate homographic ranges.*

Here I and J' coincide at O.

Ex. 2. *If I and J' be the vanishing points of the homographic ranges $(ABCP\ldots) = (A'B'C'P'\ldots)$, show that*

(*a*) $AP:AI::A'P':J'P'$;

(*b*) $AP/BP \div A'P'/B'P' = AI/BI$.

Ex. 3. *If $AB = A'B'$, show that $AI = -B'J'$.*

Ex. 4. *If O, A, B be fixed points on the fixed line OAB, and O, A', B' be fixed points on the line $OA'B'$ which may have any direction in space, show that the meet of AA' and BB' describes a sphere.*

Let AA' and BB' meet at P. Draw parallels through P to $A'B'$ and AB to meet AB at I and $A'B'$ at J'. Then I and J' are known points on the lines, being the vanishing points of the ranges determined by $(OAB) = (OA'B')$. Hence I is a fixed point, and $IP = OJ'$ is a fixed length.

3. Take any two origins U and V' on the lines of the ranges. Then $IP = UP - UI = x - a$, say;

and $J'P' = V'P' - V'J' = x' - a'$, say.

Hence we get $(x-a)(x'-a') = \text{constant}$,

or $xx' - a'x - ax' + aa' = \text{constant}$,

a relation of the form $kxx' + lx + mx' + n = 0$.

Hence *the distances x and x' of corresponding points in two homographic ranges from any fixed points on the lines of the ranges are connected by a relation of the form*

$$kxx' + lx + mx' + n = 0,$$

where k, l, m, n are constants.

Conversely, *if the distances be connected by this relation, the points generate homographic ranges.*

For if $kxx' + lx + mx' + n = 0$,

$$\text{then} \quad k\left(x + \frac{m}{k}\right)\left(x' + \frac{l}{k}\right) = \frac{lm}{k} - n,$$

or $IP \cdot J'P' =$ constant, where $m/k = IU$ and $l/k = J'V'$.

This reasoning fails if I and J' are at infinity; which is the case discussed in the next article.

Notice that the relation $kxx' + lx + mx' + n = 0$ contains really three arbitrary constants, viz. the ratios of l, m, n to k. This is as it should be, for an homography is known if three pairs of corresponding points are given; and then, substituting the values of x and x' of these points, we have three equations to determine these ratios.

Ex. *If we take V' at U' (the point corresponding to U), show that* $UI/UP + U'J'/U'P' = 1$.

4. Two ranges are said to be similar when the corresponding points divide the lines on which the ranges lie similarly. Hence if the ranges $ABC \ldots$ and $A'B'C' \ldots$ are similar we have $AB : BC : CD : \ldots :: A'B' : B'C' : C'D' : \ldots$.

Similar ranges are homographic ranges with the points at infinity corresponding; and, conversely, two homographic ranges in which the points at infinity correspond are similar.

For if the ranges $AB \ldots P \ldots$ and $A'B' \ldots P' \ldots$ are similar, we have $AB : BP :: A'B' : B'P'$. Hence

$$\frac{AB}{BP} \div \frac{A\Omega}{\Omega P} = \frac{A'B'}{B'P'} \div \frac{A'\Omega'}{\Omega'P'}$$

$$\text{or } (AP, B\Omega) = (A'P', B'\Omega').$$

Hence P and P' are corresponding points in the homographic ranges determined by $AB\Omega$, $A'B'\Omega'$. Hence P and P' describe homographic ranges in which Ω and Ω' are corresponding points.

Conversely, if $(ABC \ldots P \ldots \Omega) = (A'B'C' \ldots P' \ldots \Omega' \ldots)$ we have $(AP, B\Omega) = (A'P', B'\Omega')$. Hence

$$AB : BP :: A'B' : B'P'.$$

Hence the ranges are similar.

Notice that if Ω and Ω' correspond, I and J' are both at infinity. For if Ω' corresponds to Ω, J' is Ω'; so for I.

In similar ranges, the relation is of the form $lx + mx' + n = 0$; *and, conversely, if the relation is of the form* $lx + mx' + n = 0$, *the ranges are similar.*

The general relation $kxx' + lx + mx' + n = 0$ may be written

$$k + \frac{l}{x'} + \frac{m}{x} + \frac{n}{xx'} = 0.$$

Hence if the ranges are similar, i.e. if Ω and Ω' correspond, i.e. if when $x = \infty$, $x' = \infty$, we have $k = 0$; and the relation reduces to $lx + mx' + n = 0$. And, conversely, if the relation is of the form $lx + mx' + n = 0$, we have $x' = \infty$ when $x = \infty$. Hence Ω and Ω' correspond, i.e. the ranges are similar.

Notice that in this case, in the construction of § 1, AA', BB', ... are parallel and O is at infinity.

Ex. *Obtain the Cartesian equation of a straight line.*

The feet of the ordinate and abscissa generate ranges similar to the range of points on the line. Hence $lx + my + n = 0$.

5. The relation $kxx' + lx + mx' + n = 0$ is called a (1, 1) relation because if we are given x, then x' has one and only one value, and if we are given x', then x has one and only one value. Hence we have proved that if two ranges are homographic, the distances of two corresponding points P and P' from fixed points U and V' on the lines of the ranges are connected by a (1, 1) relation; and, conversely, if the distances are connected by a (1, 1) relation, then the ranges are homographic.

Hence if P and P' generate homographic ranges on fixed lines, P and P' are in (1, 1) correspondence, i.e. if P is given, then P' is known uniquely and if P' is given, then P is known uniquely. The question now arises—does a (1, 1) correspondence between two points necessarily involve that these points generate homographic ranges? The answer to this question is contained in the following theorem.

**If the variable points P and P' on the fixed lines l and l' are connected by a series of linear constructions involving only the finding of the intersection of known lines, the connector of known points, the second intersection of a known line with a known*

conic, one intersection being given, the second tangent from a known point to a known conic, one tangent being given, the finding of the polar of a known point for a known conic or of the pole of a known line for a known conic or finally the unique construction of a conic from given conditions, and if when P is given, P' is known uniquely and when P' is given, P is known uniquely, then P and P' generate homographic ranges.

For taking fixed points U and V' on l and l', let $UP = r$ and $V'P' = r'$. Let us work at these constructions by Analytical Geometry and thus obtain the relation between r and r'.

If P is known, r is known. Hence the result of the first construction will be to obtain the coordinates of a point or the coefficients of the equation of a line or of a curve in terms of r. Also as all the specified constructions have unique solutions, it follows that these coordinates or coefficients will be rational functions of r. For if a radical of any sort entered, the solution would not be unique; and the above constructions do not introduce transcendental functions. Now work at the next construction. As above the new coordinates or coefficients will be rational functions of the old coordinates and coefficients, and therefore rational functions of r. So we can proceed until we finally see that r' is a rational function of r.

Also this rational function of r cannot contain r^2 or any higher power of r; for if so, when P' is known, and therefore r', there would be (by the solution of the quadratic or higher equation in r) more than one value of r and therefore more than one position of P. Hence we must have

$$r' = (ar + b)/(cr + d);$$

which is of the form

$$krr' + lr + mr' + n = 0,$$

and proves that P and P' generate homographic ranges.

Such a (1, 1) correspondence may be called a rational or homographic (1, 1) correspondence.

This is not the complete theory of rational (1, 1) correspondence; but it is sufficient for the purposes of this

treatise. As a matter of fact, it is easy to extend the above enunciation and proof to any rational curves.

The reader should notice that it is not true that two points P and P' on fixed lines l and l' which are connected by *any* (1, 1) correspondence generate homographic ranges.

For instance, if we put $r = x + yi$ and $r' = x - yi$, P and P' are connected by a (1, 1) correspondence; but this is not homographic, for no algebraical relation can be obtained between r and r'.

Again if V is a point on a given conic and if VP and VP' cut the conic again in Q and Q' which are the ends of a chord of given length whose ends slide on the arc of the conic, P and P' are connected by a (1, 1) correspondence which is not homographic.

The reason in both cases is the same, viz. that r' cannot be rationally expressed in terms of r.

Also it is clear from the proof, that we may add to the above list of constructions, any in which the final coordinates and coefficients of the construction can be expressed rationally in terms of the initial coordinates and coefficients.

To avoid each time quoting the above list of constructions, we may say that each of them is *a rational construction.* In every case the final appeal as to whether a construction is rational or not must be to Analytical Geometry.

The simplest case of a rational (1, 1) correspondence between P and P' is when a relation is given connecting P and P' which evidently becomes a rational algebraical relation connecting x and x' when we put $x - a$ for AP and so on. In this case, as the relation is evidently rational, we have only to verify that it is (1, 1), i.e. that P' is given uniquely when P is known and that P is given uniquely when P' is known.

Ex. 1. *If O, U, V' are fixed points, and if points P and P' are taken on OU and OV' such that*

$$a \,.\, UP/OP + b \,.\, V'P'/OP' = 0$$

where a and b are constants, show that PP' passes through a fixed point on UV'.

If P is given, UP/OP is known, and hence from the given relation $V'P'/OP'$ is known uniquely and therefore P'; so P is given uniquely by P'. Also by putting $UP = x - u$ and so on we see that the given relation is rational. Hence it must reduce to a (1, 1) relation. Hence P and P' generate homographic ranges. Also when P is at O, $OP = 0$, and hence we see by rationalizing the given relation that $OP' = 0$; hence P' is at O. Hence O corresponds to itself. Hence PP' passes through a fixed point. Also when P is at U, $UP = 0$, hence $V'P' = 0$, hence P' is at V'. Hence UV' is one position of PP'. Hence the fixed point is on UV'.

Here it is easy to verify directly that the given relation is of the form $kxx' + lx + mx' + n = 0$.

Ex. 2. *Obtain the envelope of PP' when the given relation is $a.UP + b.V'P' = 0$ where U and V' are fixed points on the parallel lines UP and $V'P'$.*

***6.** *If the rays VP and $V'P'$ which pass through the fixed points V and V' are connected by a (1, 1) rational construction, then VP and $V'P'$ generate homographic pencils.*

Let VP and $V'P'$ cut two fixed lines l and l' at P and P'. Then the construction connecting P and P' is rational; for we have merely added to the given construction the finding of the intersection of VP with l and of $V'P'$ with l'. It is also (1, 1); for when P is given, VP is given and therefore $V'P'$ by hypothesis and therefore P', all uniquely, and so P when P' is given. Hence P and P' are connected by a rational (1, 1) correspondence and therefore generate homographic ranges. Also $V(P) = (P) = (P') = V'(P')$; hence VP and $V'P'$ generate homographic pencils.

It is interesting to note that the (1, 1) relation in the case of pencils is *not* $k\theta\theta' + l\theta + m\theta' + n = 0$ where θ and θ' are the circular measures of the angles which corresponding rays make with fixed lines through the vertices. For if UP is given, θ is not given; in fact if $\theta = a$ is one value of θ, the general value is $a \pm 2r\pi$, r being any integer. Hence the above relation is *not* a (1, 1) relation.

In exactly the same way by taking a section of the pencil we show that *if the points of a range and the rays of a pencil*

are connected by a rational (1, 1) *construction, the range and the pencil are homographic.*

Ex. 1. *Show that the* (1, 1) *relation of two pencils is*

$$k\tan\theta\tan\theta' + l\tan\theta + m\tan\theta' + n = 0.$$

Measure the pencils by ranges on perpendiculars to the initial lines.

Ex. 2. *Show that the relation*

$$k\sin\theta\sin\theta' + l\sin\theta + m\sin\theta' + n = 0$$

is not (1, 1).

Ex. 3. *If O, O', A, A', U, V' are fixed points of which O, A, A', O' are collinear, and if points P and P' are taken on AU and $A'V'$ such that $a\,.\,UP/AP + b\,.\,V'P'/A'P' = c$ where a, b, c are constants, show that the locus of the intersection of OP and $O'P'$ is a line.*

As before P and P' are connected by a rational (1, 1) relation and therefore generate homographic ranges. Hence OP and $O'P'$ generate homographic pencils. Hence OP and $O'P'$ will meet on a line if OO' and $O'O$ correspond, i.e. if when P is at A, P' is at A'. But when P is at A, $AP = 0$. Hence rationalizing the given relation we see that $A'P' = 0$, i.e. P' is at A'.

It is easily verified in this case that the given relation is of the form $kxx' + lx + mx' + n = 0$ by rationalizing and then putting for $UP = x - u$ and so on.

Ex. 4. *Show that Ex. 3 holds also for the relation*

$$a/AP + b/A'P' = c.$$

Ex. 5. *If $c = 0$ in Ex. 3, one point on the locus is*

$$(OU;\ O'V').$$

Common points of two homographic ranges on the same line.

7. Suppose corresponding points in two ranges on the same line to be connected by the relation

$$k\,.\,UP\,.\,V'P' + l\,.\,UP + m\,.\,V'P' + n = 0.$$

For the origins U and V' we can take the same point U on the line. The equation becomes

$$k\,.\,UP\,.\,UP' + l\,.\,UP + m\,.\,UP' + n = 0.$$

Now if P correspond to itself, P must coincide with P'.

Hence the equation giving the self-corresponding or *common points* of the two ranges is

$$k \,.\, UP^2 + (l+m)\, UP + n = 0.$$

Hence every two homographic ranges on the same line have two common points, real, coincident, or imaginary.

A graphic construction of the common points will be found in XVI. 6.

If E and F be the common points of the homographic ranges $(ABC \ldots)$ and $(A'B'C' \ldots)$, then

$$(EFAA') = (EFBB') = (EFCC') = \ldots .$$

For $(EFABC \ldots) = (EFA'B'C' \ldots)$. Hence

$$(EF,\, AB) = (EF,\, A'B').$$

Hence $\quad EA/AF \div EB/BF = EA'/A'F \div EB'/B'F.$

Hence $\quad (EF,\, AA') = (EF,\, BB').$

If $(EF,\, PP')$ be constant, then P, P' generate homographic ranges of which EF are the common points.

For $(EF,\, PP') = \lambda$ gives $EP/PF = \lambda \,.\, EP'/P'F$ which is a rational (1, 1) relation from which $P' = E$ if $P = E$; so for F.

Ex. 1. *If $ABC \ldots$, $A'B'C' \ldots$ be homographic ranges on the same line, and if P', Q be the points corresponding to the point $P (= Q')$ according as it is considered to belong to the first range or the second, show that P', Q generate homographic ranges whose common points are the same as those of the given ranges.*

The range generated by P' is homographic with the range generated by P, i.e. by Q', and this is homographic with the range generated by Q. Hence range P' = range Q.

Again, suppose P is a common point of the given ranges; then P' coincides with P, i.e. P' coincides with Q'; hence P coincides with Q, i.e. P' coincides with Q, i.e. P is a common point of the derived ranges.

Ex. 2. *Determine the point X, given the value of*

$$AX \,.\, A'X \div BX.$$

Consider the relation $AP \,.\, A'P' \div BP = \lambda$. This is a rational (1, 1) relation. Hence P and P' generate homographic ranges of which the common points give the positions of X.

Ex. 3. *Show that the homographic relation can be thrown into the form* $EP \,.\, FP' + IP \,.\, PP' = 0.$

Taking any origin, we see that this is a rational (1, 1) relation. Also when $P=E$, $P'=P=E$; when $P=F$, $P'=P=F$. Again, since $EP.FP/IP+PP'=0$, we see that when $P=I$, $P'I=P'P=\infty$, i.e. $P'=\Omega'$. Hence E, F, I correspond to E, F, Ω'. Hence the homography is the given homography.

Ex. 4. *Show that another form is* $EP.FP'=EI.PP'$.

Ex. 5. *Show that* EF *and* IJ' *are bisected at the same point.*
In $IP.J'P'=IA.J'A'$ put $P=P'$.

8. *If one of the common points of two homographic ranges* $(ABC\ldots)$ *and* $(A'B'C'\ldots)$ *on the same line is at infinity, the ranges are similar; and, conversely, if two ranges are similar, one of the common points is at infinity.*

For in this case the points at infinity correspond; and the above is proved in § 4.

If both common points are at infinity, the ranges are superposable; and, conversely, in superposable ranges both common points are at infinity.

The common points of the homography of which the relation is $kxx'+lx+mx'+n=0$ are given by

$$kx^2+lx+mx+n=0 \quad \text{or} \quad k+(l+m)x^{-1}+nx^{-2}=0.$$

Hence if both roots are $x=\infty$, we must have $k=0$ and $l+m=0$. Hence the relation becomes $l(x-x')+n=0$. Hence $PP'=x'-x$ is constant, i.e. the ranges are superposable.

Conversely, if the ranges are superposable, we have $x-x'=c$. Hence as above both roots of the quadratic giving the common points are infinite; i.e. both common points are at infinity.

Ex. 1. *If one of the common points of two homographic ranges on the same line be at infinity, the other, E, is given by*

$$EA:EA'::BA:B'A'.$$

Ex. 2. *If* $AB/A'B'=BC/B'C'=\ldots=-1$, *show that one common point is at infinity, and that the other bisects all the segments* AA', BB', CC', ...

Ex. 3. *Two ranges whose common points coincide can be projected into two ranges whose corresponding segments are equal.*

Common rays of two homographic pencils having the same vertex.

9. *In any two homographic pencils having the same vertex, two rays exist, each of which corresponds to itself.*

Let the pencils be $V(ABC\ldots) = V(A'B'C'\ldots)$. Suppose a line to cut the pencils in the ranges $(abc\ldots) = (a'b'c'\ldots)$, a being on VA, and so on. Then if VA and VA' coincide, a and a' will coincide. Hence if e and f be the self-corresponding points of the ranges $(abc\ldots)$ and $(a'b'c'\ldots)$, Ve and Vf are the self-corresponding or *common rays* of the pencils $V(ABC\ldots)$ and $V(A'B'C'\ldots)$.

If VE, VF are the common rays of the two homographic pencils $V(ABC\ldots)$ and $V(A'B'C'\ldots)$, then

$$V(EF, AA') = V(EF, BB') = \ldots;$$

and, conversely, if $V(EF, PP')$ is constant, VE and VF being fixed lines, then VP and VP' generate homographic pencils of which VE and VF are the common rays.

For consider a section of the pencils; and see § 7.

Ex. 1. *Find a point on a given line through which shall pass a pair of corresponding lines of two given homographic pencils.*

Either of the common points of the homographic ranges determined on the line by the pencils.

Ex. 2. *If VA, $V'A'$ generate homographic pencils at V and V', show that in two positions VA is parallel to $V'A'$; and that any transversal in either of these directions is cut by the two pencils proportionally.*

For without altering the directions of the rays, superpose V' on V.

Ex. 3. *Two given homographic pencils $V(abc\ldots)$ and $V'(a'b'c'\ldots)$ meet a line in the points $ABC\ldots$ and $A'B'C'\ldots$; determine the position of the line so that $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, &c.*

Suppose the line drawn. Since $(\Omega ABC\ldots) = (\Omega A'B'C'\ldots)$, the line must be parallel to one or other of the pairs of

corresponding parallel rays. Let it meet the other two corresponding parallel rays in O, O'. Draw $V'S$ parallel to the line to meet VO in S. Then

$$SO = V'O', \quad OA = O'A', \text{ and } \angle SOA = \angle V'O'A'.$$

Hence SA is parallel to $V'A'$.

Hence the construction—Take the corresponding rays Vy, $V'y'$ which are parallel, and also the corresponding rays Vz, $V'z'$ which are parallel. Let Vz meet $V'y'$ in S, and through S draw SA parallel to $V'a'$ to meet Va in A. Through A draw $ABC\ldots$ $A'B'C'\ldots$ parallel to $V'S$. This line satisfies the required condition.

For Vy, $V'y'$ meet the line in the same point Ω at infinity. Hence $(\Omega OAB) = (\Omega O'A'B')$. Hence

$$OA : OB :: O'A' : O'B'.$$

But $OA = O'A'$ by construction. Hence $OB = O'B'$. Hence $AB = A'B'$, and so on.

Hence there are two such lines, one parallel to each of the lines Vy, Vz.

Ex. 4. *Given any two homographic pencils, one can be moved parallel to itself so as to be in perspective with the other.*

***10.** *If I, J' correspond to the points at infinity in two homographic ranges on the same line, and O bisect IJ', and O' be the point corresponding to O, then the common points E, F are given by*

$$OE^2 = OF^2 = OJ'.OO'.$$

For $$(O\Omega, IE) = (O'J', \Omega'E),$$

where Ω or Ω' is the point at infinity upon the line.

$$\therefore \quad \frac{OI}{I\Omega} \cdot \frac{E\Omega}{OE} = \frac{O'\Omega'}{\Omega'J'} \cdot \frac{EJ'}{O'E}.$$

But $$E\Omega \div I\Omega = 1 \text{ and } O'\Omega' \div \Omega'J' = -1,$$

$$\therefore\ OI.O'E + OE.EJ' = 0.$$

Take O as origin, $\therefore\ OI(OE - OO') + OE(OJ' - OE) = 0,$

but $OI = -OJ'$, $\therefore\ -OJ'(OE - OO') + OE(OJ' - OE) = 0,$

$$\therefore\ OE^2 = OJ'.OO'; \text{ so } OF^2 = OJ'.OO'.$$

Hence *the two common points are equidistant from O*; therefore one is as far from I as the other is from J'.

Notice that $(EF, O'J')$ *is harmonic.*

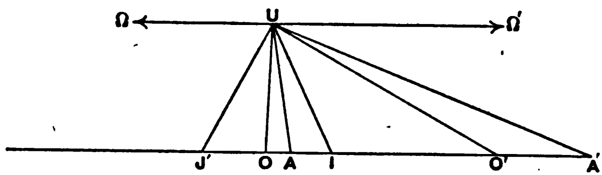
***11.** *If the common points be imaginary, then the ranges*

$(ABC\ldots)$ and $(A'B'C'\ldots)$ subtend at two points in the plane of the paper superposable pencils.

For if E and F are imaginary, since $OE^2 = OJ'.OO'$, we see that OJ' and OO' have different signs, i.e. O lies between O' and J'. On a perpendicular to the line AA' through O take OU, such that $OU^2 = OJ'.O'O$. Two such points can be taken one on each side of the line AA'.

Then the pencils subtended at U are superposable.

Since I corresponds to the point Ω' at infinity, the ray $U\Omega'$ is parallel to AA'; so the ray $U\Omega$ corresponding to



UJ' is parallel to AA'. Now since $UO^2 = J'O.OO'$ it follows that $J'UO'$ is a right angle.

Hence $$\angle\Omega UJ' = \angle UJ'O = \angle OUO'$$
$$= \angle UIJ', \text{ since } J'O = OI$$
$$= \angle IU\Omega'.$$

Hence the pencil $U(\Omega OI)$ can be superposed to the pencil $U(J'O'\Omega')$ by turning it through the angle $\Omega UJ'$. After the rotation three rays of the pencils $U(\Omega OIABC\ldots)$ and $U(J'O'\Omega' A'B'C'\ldots)$ coincide; hence every ray of one pencil coincides with the corresponding ray of the other pencil, i.e. the pencils are superposed.

Also *these are the only points satisfying the condition.* For we must have $\angle\Omega UJ' = \angle IU\Omega'$. Hence $UJ' = UI$. But $OJ' = OI$. Hence OU is perpendicular to AA'. Also $\angle OUO' = \angle\Omega UJ' = \angle UJ'O$. Hence $J'UO'$ is a right angle. Hence $OU^2 = OJ'.O'O$. Hence these are the only positions of U.

Notice that the points U give solutions of the problem—*Given, on one line, two homographic ranges $(ABC\ldots)$ and*

$(A'B'C'...)$ of which the common points are imaginary, find a point at which the segments AA', BB', CC', ... subtend equal angles.

Ex. *Determine a point at which three given collinear segments subtend equal angles.*

***12.** *Two homographic pencils with the same vertex whose common rays are imaginary can be placed in perspective with two superposable pencils.*

For let any line cut the given pencils in $ABC...$ and $A'B'C'...$. In a plane not that of the pencils construct the point U at which AA', BB', ... subtend equal angles. Take the vertex of projection on the line joining U to the vertex V of the given pencils; and take the plane of projection parallel to UAA'. Then the projection of VA is parallel to UA, and of VA' to UA'. Hence the projection of the angle AVA' is equal to the angle AUA'; so for the other angles. Hence the angles AVA', BVB', CVC', ... project into equal angles.

X

1. If one of two copolar triangles be rotated about the axis of homology, show that the centre of homology describes a circle.

2. If PM, PM' drawn in given directions from P meet given lines OM and OM' at M and M', so that

$$a.PM + b.PM' = c$$

where a, b, c are constants, show that P moves on a straight line.

3. If the perpendiculars from A, B, C on a line be connected by the relation $ap + bq + cr = 0$ where a, b, c are constants, show, geometrically, that the line passes through a fixed point.

4. Two homographic ranges are taken on the same line, and to the point called P or Q' the corresponding points are P' and Q. If $(PX, P'Q)$ is harmonic, then (PX, EF) is harmonic, E and F being the self-corresponding points of the given ranges.

5. If $(EFABC\ldots) = (EFA'B'C'\ldots)$
$$= (EFA''B''C''\ldots) = \ldots,$$
show that
$$(EFAA'A''\ldots) = (EFBB'B''\ldots) = (EFCC'C''\ldots) = \ldots.$$

6. If, in two homographic ranges, A', A'' in the second range correspond to A, A' in the first, show that if the self-corresponding points coincide at E, then $(E\,A', AA'')$ is harmonic.

7. If $(ABXX') = (CDXX')$, A, B, C, D, X, X' being collinear points, show that either X and X' generate homographic ranges or X and X' coincide.

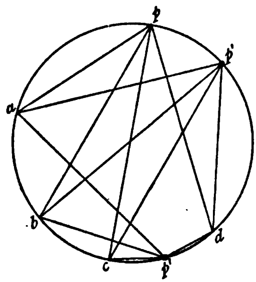
8. If O bisects EF and O' corresponds to O, show that the homographic relation can be thrown into the form
$$OP\,.\,OP' - OI\,.\,PP' + OI\,.\,OO' = 0.$$

CHAPTER XI

ANHARMONIC PROPERTIES OF POINTS ON A CONIC

1. We have already shown in IX. 8 that the projection of a range of four points is homographic with the range, and in IX. 9 that the projection of a pencil of four lines is homographic with the pencil. We shall now proceed to investigate certain properties of a conic by proving the corresponding properties of the circle of which the conic is by definition the projection.

2. *Four fixed points on a conic subtend at a variable fifth point on the conic a constant cross ratio.*



Let the four fixed points on the conic be $ABCD$ and the variable point P. Let A, B, C, D, P be the projections of the points a, b, c, d, p on the circle of which the conic is the projection. Now, in the circle, $abcd$ subtend the same cross ratio at every point on the circle. For take any two points p and p' on the circle. Then

$$p(ab, cd) = \frac{\sin apc}{\sin cpb} \div \frac{\sin apd}{\sin dpb} = \frac{\sin ap'c}{\sin cp'b} \div \frac{\sin ap'd}{\sin dp'b} = p'(ab, cd).$$

For in all cases the angle apb is equal to the angle $ap'b$ or its supplement; and so for the other angles. Hence $P(ABCD) = p(abcd)$ by projection $= p'(abcd) = P'(ABCD)$ by projection. Hence $ABCD$ subtend the same cross ratio at every point P on the conic.

The cross ratio subtended by the points (AB, CD) on a conic at any point on the conic is called the *cross ratio of the points* (AB, CD) on the conic.

Notice that, making P coincide with A, the cross ratio of (AB, CD) is equal to $A\,(AB, CD) = A\,(TB, CD)$, where AT is the tangent to the conic at A.

Another proof is obtained from the fact that the two pencils $p\,(abcd)$ and $p'\,(abcd)$ in the case of the circle are superposable in all cases. This proof is tedious on account of the number of different cases which arise according to the position of p'.

Another proof is by (1, 1) correspondence; for which see § 4 end.

An important case is that in which four points subtend a harmonic pencil at every point of a conic. Such points are called *harmonic points*; if $P\,(AA', BB')$ is harmonic, then A, A' and B, B' are called harmonic pairs of points on the conic.

If A, A' and B, B' are harmonic pairs of points on a conic, then AA' and BB' are conjugate lines; and, conversely, if AA' and BB' are conjugate chords of a conic, then A, A' and B, B' are harmonic pairs of points on the conic.

Let AA' and the tangent at A cut BB' at M and T. Then $-1 = A\,(AA', BB') = (TM, BB')$. Hence the polar of T passes through M; and also through A, and hence is AA'. Hence AA' and BB' are conjugate.

Conversely if AA' and BB' are conjugate, the pole of AA' is on BB' and also on AT; and hence is T. Hence

$$-1 = (TM, BB') = A\,(AA', BB').$$

Hence (AA', BB') is harmonic.

Ex. 1. *A tangent to an ellipse meets the auxiliary circle in ZZ'; show that the cross ratio of the four points (AA', ZZ') on the circle is $(1-e) \div (1+e)$.*

Consider the pencil at the point opposite to Z'.

Ex. 2. *Prove that the cross ratio (AB, CD) of the four points A, B, C, D on a circle is $AC/CB \div AD/DB$, AC being the length of the line joining A to C.*

For $\sin APC = AC \div 2R$.

Ex. 3. *Two conics α and β touch at B and C. Through A, the meet of the common tangents, is drawn a line meeting α in*

P, Q. BQ, BP meet β in V, U. Show that VU passes through A.

For $(BC, UV) = B(BC, UV) = B(AC, PQ) = -1$.

Ex. 4. *If AA', BB' be pairs of harmonic points on a circle, show that $AA' . BB' = 2 . AB . A'B' = 2 . AB' . BA'$.*

Use Ex. 2.

Ex. 5. *Any diameter of a parabola meets the tangent at Q in T, the curve in P, and any chord QQ' in R; show that*

$$TP : PR :: QR : RQ'.$$

For $Q(QPQ'\Omega) = \Omega(QPQ'\Omega)$, Ω being the point at infinity on the parabola.

Ex. 6. *A variable point P on a conic is joined to the fixed points L, M on the conic; show that the angle LPM is divided in a constant cross ratio by parallels through P to the asymptotes.*

***3.** Pappus's theorem. *If from any point P on a conic perpendiculars α, β, γ, δ be drawn on the lines AB, BC, CD, DA joining fixed points $ABCD$ on the conic, then $\alpha . \gamma = k . \beta . \delta$, where k is independent of the position of P.*

For $P(AC, BD) = \sin APB . \sin DPC \div \sin BPC . \sin APD$.

But $PA . PB \sin APB = \alpha . AB$, and so on.

Hence $\alpha . \gamma . AB . DC \div \beta . \delta . BC . AD = P(AC, BD)$ is constant, i.e. $\alpha . \gamma = k . \beta . \delta$.

If the conic is a circle, $AB = 2R \sin APB$, and so on. Hence $k = 1$.

Ex. 1. *If the perpendiculars let fall from any point on a conic on the sides of an inscribed polygon of an even number of sides be called* 1, 2, 3, ..., $2n$, *show that*

$$1 . 3 . 5 . \ldots (2n-1) \div 2 . 4 . 6 . \ldots . 2n$$

is constant.

Suppose the theorem holds for $2n-2$ sides. Then

$$1 . 3 . 5 . \ldots (2n-3) = k . 2 . 4 . 6 . \ldots (2n-4) x.$$

And by the above theorem $(2n-1)x = k'(2n-2)(2n)$.
Multiplying, $1 . 3 . 5 . \ldots (2n-1) = k'' . 2 . 4 . 6 . \ldots . 2n$.
Hence by Induction.

Ex. 2. *The product of the perpendiculars from any point on a conic on the sides of any inscribed polygon varies as the product of the perpendiculars on the tangents at the vertices.*

Make the alternate sides in Ex. 1 of zero length.

Ex. 3. *The product of the perpendiculars from any point on a conic on two fixed tangents is proportional to the square of the perpendicular on the chord of contact.*

Ex. 4. *The product of the perpendiculars from any point on a hyperbola on two fixed lines, one parallel to each asymptote, is proportional to the perpendicular on the chord joining the points in which the lines meet the curve.*

For $\alpha . \gamma \div \beta . \delta = \alpha' . \gamma' \div \beta' . \delta'$ and $\alpha = \alpha'$.

Ex. 5. *The product of the perpendiculars from any point on a parabola on two fixed diameters is proportional to the perpendicular on the chord joining the points in which the lines meet the curve.*

4. ***Any number of fixed points on a conic subtend homographic pencils at variable points on the conic.***

Let the fixed points be $A, B, C, D, \ldots$ and take two other points P, Q on the conic; we have to prove that $P(ABCD\ldots) = Q(ABCD\ldots)$. This follows at once from the fact that $P(ABCD) = Q(ABCD)$, where $ABCD$ are any four of the fixed points.

Or thus. To show that $P(ABC\ldots) = Q(ABC\ldots)$, suppose the ray PR of the first pencil given, then R is given as the intersection (other than P) of PR and the conic; hence QR is given uniquely. So if QR is given, PR is given uniquely. Hence PR and QR are connected by a (1, 1) construction which is clearly rational. Hence PR and QR generate homographic pencils. Hence $P(ABC\ldots) = Q(ABC\ldots)$.

Hence as a particular case, $P(ABCD) = Q(ABCD)$, which is the result of § 1.

Notice that in this proof we assume nothing about a conic except that it is of the second order, i.e. that a line meets it in two points.

Ex. 1. *P, U, V are points on a hyperbola, P being variable; show that the lines PU and PV intercept on either asymptote a constant length.*

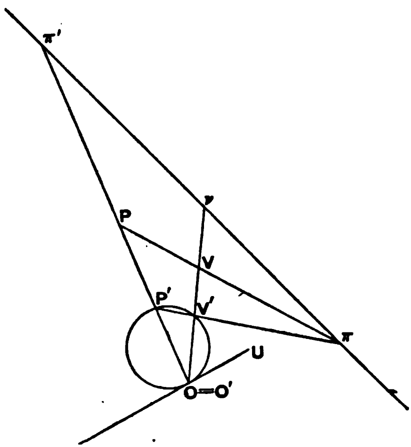
Instead of the asymptote consider at first a chord LM of the conic, and let PU, PV cut LM in p and p'. Then $(p) = U(P) = V(P) = (p')$. And the common points of the homographic ranges (p) and (p') are seen, by taking P at L and M, to be L and M. Hence in the given case the common points coincide at infinity; hence pp' is constant.

Ex. 2. *Through a fixed point are drawn lines parallel to the rays of the pencils subtended at two points on a parabola by the other points on the parabola; show that corresponding lines cut off on a fixed diameter a constant length.*

Join the ranges determined on the line at infinity to the fixed point and proceed as above.

***5.** *The locus of the meets of corresponding rays of two homographic pencils, at different vertices and not in perspective, is a conic which passes through the vertices.*

Let the pencils be $O(PQR...)$ and $V(PQR...)$. Then we



have to prove that the locus of the points $PQR...$ is a conic through O and V. Since the pencils are not in perspective, corresponding to the ray VO in the V pencil, we shall have some ray OU, say, in the O pencil which does not coincide with VO. Draw any circle touching OU at O. Let this circle cut OV in V', OP in P', and so on.

We are given that $V(PQ...)$ and $O(PQ...)$ are homographic pencils. Also OU in the pencil at O corresponds to VO in the pencil at V. Hence $V(OPQ...)$ and $O(UPQ...)$ are homographic. Again $O(UPQ...)$ is homographic with $O(OP'Q'...)$; for the ray OU is the tangent to the circle at O and therefore joins O to the point consecutive to O on the

circle. Again by the homographic property of a circle $O(OP'Q'...)$ is homographic with $V(OP'Q'...)$. Hence

$$V(OPQ...) = O(UPQ...) = O(OP'Q'...) = V(OP'Q'...).$$

Hence the two pencils $V(OPQ...)$ and $V'(OP'Q'...)$ are homographic. And they have a common ray, viz. $VV'O$. Hence they are in perspective. Hence all the points $(VP;\ V'P')$, $(VQ;\ V'Q')$, ... lie on a line, viz. the axis of perspective. Let $(VP;\ V'P')$ be called π; and let the axis meet OV in ν and OP in π'; so for $Q, R, \ldots$.

Now rotate the figure of the circle out of the original plane about the axis $\pi\pi'\ldots$; and let O' be the new position of O. Then the triangles OPV and $O'P'V'$ are coaxal; for OP and $O'P'$ meet in π', and OV and $O'V'$ meet in ν, and PV and $P'V'$ meet in π. Hence these triangles are copolar, i.e. OO', PP', VV' meet in a point. Hence PP' passes through a fixed point, viz. the meet of OO', VV'. Hence the figure $OVPQR\ldots$ is the projection of the figure $O'V'P'Q'R'\ldots$. But the latter figure is a circle; hence the locus of P is the projection of a circle, i.e. is a conic. Also, since the circle passes through V' and O', the conic passes through V and O.

If, however, the pencils are in perspective, then the ray VO corresponds to the ray OV, and we know by IX. 16 that corresponding rays meet on the axis of perspective. Hence a part of the locus of $P, Q, \ldots$ is the axis of perspective. Another part is the line joining the vertices V and O of the pencils; for if we take any point P on VO, then VP and OP are corresponding rays of the pencils, since VO corresponds to OV. Hence the conic degenerates into the axis of perspective and the line joining the vertices.

Notice that in the general case when VO does not correspond to OV, *the ray corresponding to OV is the tangent at V to the locus*; for when P is very near V, OP is very near OV.

Ex. 1. *Project any five points which lie in the same plane, but no three of which lie on the same line, into points on the same circle.*

Let O, V, P, Q, R be the points; and consider the pencils determined by $O(PQR) = V(PQR)$.

Ex. 2. *Any two conics can be placed in perspective by first placing them so as to touch one another.*

6. *One, and only one, conic can be drawn through five given points.*

Let the five points be A, B, C, D, E. Take A and B as vertices. Through A draw any ray AP, and let BQ be such that $A(CDEP) = B(CDEQ)$. Then the rays AP and BQ generate homographic ranges of which AC and BC, AD and BD, AE and BE are corresponding rays. Hence the locus of the meet R of the rays AP and BQ is a conic through $ABCDE$. Hence a conic can be drawn through $ABCDE$.

Also only one conic can be drawn through $ABCDE$. For the other point R, in which any ray AP cuts a conic through $ABCDE$, is given by the relation $A(CDER) = B(CDER)$. Hence every ray through A cuts all conics through $ABCDE$ in the same point, i. e. all the conics coincide.

See also VI. 8.

It is assumed in the above proof that no three of the points lie on a straight line. *If three of the points are collinear, the conic through the five points degenerates into a pair of straight lines,* viz. the line through the three points and the line through the other two points.

The locus of points at which four given points subtend a constant cross ratio is a conic through the given points.

Let the points $ABCD$ subtend the same cross ratio at $E, P, Q, R\dots$. Then, taking E and P as vertices, since

$$E(ABCD) = P(ABCD),$$

we know that $ABCDEP$ lie on a conic. Hence the locus of P is the conic drawn through the five fixed points A, B, C, D, E.

Ex. 1. *Any four points A, B, C, D are taken, and M is the middle point of AC; BQ a parallel to AC cuts DM in Q, and DP a parallel to AC cuts BM in P; show that $ABCDPQ$ lie on a conic.*

For $P(AC, BD) = Q(AC, BD) = -1$ on AC.

Ex. 2. *If P, Q, A, B, C, D be six points on a conic; show*

that the meets of PA and QB, of PB and QA, of PC and QD, and of PD and QC lie on a conic through PQ.

Ex. 3. *Given in position two pairs of conjugate diameters of a conic and a point P on the conic, to construct it.*

Ex. 4. *What is the conic which passes through five given points,* (i) *four,* (ii) *five of which are collinear?*

7. *Every two conics cut in four points.*

Two conics cannot cut in more than four points; for if they have five points in common, they must coincide. Also we see that two equal ellipses laid across one another cut in four points. Hence we conclude that if two conics do not apparently cut in four points, some of the meets are imaginary or coincident. (See also XXVII. 4.)

Through four given points can be drawn an infinite number of conics.

For we can draw a conic through the four given points and any fifth point.

All conics through four given points have a common self-conjugate triangle; viz. the harmonic triangle of the quadrangle formed by the points.

Hence *any two conics have a common self-conjugate triangle* since they intersect in four points.

There are several particular cases.

If two intersections coincide so that the conics touch, let a and b in the figure of XIX. 6 coincide, then V and W coincide at a and U is the point in which the common tangent at a cuts cd; the direction of VW is known, for it passes through the fourth harmonic of U for c and d.

If three intersections coincide, so that the conics have three-point contact, let c move up to a in the above, then U also moves up to a and U, V, W all coincide with a.

If the intersections coincide in pairs so that the conics have double contact, let b move up to a and d move up to c in the figure of XIX. 6, then U is the intersection of the tangents at a and c and V and W are some points on ac; in fact any two points V and W which are harmonic with a and c will give a self-conjugate triangle UVW, as is easily verified.

If all four intersections coincide so that the conics have four-point contact, let c, d in the above move up to a, b, then U also moves up to a and so does V or W since V and W are harmonic with a, c; hence U, V coincide at a and W is any point on the common tangent at a, UV being the common polar of W.

8. *A, B, C, D are fixed points. CD meets AP in M and BP in N; find the locus of P, given that the ratio $CM:DN$ is constant. Discuss the locus when AB and CD are parallel.*

Since $CM = k \,.\, DN$, M and N generate homographic ranges on CD (see X. 4). Hence

$$A(P_1P_2\ldots) = (M_1M_2\ldots) = (N_1N_2\ldots) = B(P_1P_2\ldots).$$

Hence the locus of P is a conic through A and B.

If AB and CD are parallel, take M is at infinity, then N is also at infinity, and AM and BN coincide with AB. Hence AB and BA correspond. Hence the locus degenerates into a line and the line AB.

Ex. 1. *The locus of the vertex of a triangle, whose base is fixed, and whose sides cut off a constant length from a given line, is a conic, one of whose asymptotes is parallel to the given line.*

For when M and N are at infinity, AP is parallel to MN.

Ex. 2. *A triangle ABC is such that B and C move on fixed lines OL and OM, whilst its sides BC, CA, AB pass through fixed points P, Q, R; show that the locus of A is a conic passing through R, Q, O and through the meet of PQ and OL and through the meet of PR and OM.*

For $\quad R(A_1A_2\ldots) = (B_1B_2\ldots) = P(B_1B_2\ldots)$
$$= (C_1C_2\ldots) = Q(A_1A_2\ldots)$$

or briefly $R(A) = (B) = P(B) = (C) = Q(A)$. To verify that O, $(PQ;\ OL)$, $(PR;\ OM)$ lie on the locus, suppose A at these points and show that the construction holds.

Ex. 3. *A, A' are fixed points on a circle and the arc PP' moves round the circle; show that the locus of the intersection of AP, $A'P'$ is a conic.*

For $\quad A(P\ldots) = A(P'\ldots)$ since $\angle PAP'$ is given
$$= A'(P'\ldots).$$

Ex. 4. *A and M are fixed points, P is a variable point moving on a fixed line l, QM at right angles to PM meets PA in Q; show*

that the locus of Q is a conic. If l meet the circle on AM as diameter in B and C, show that the asymptotes of the conic are parallel to AB, AC.

Ex. 5. *A and B are fixed points, and P and Q are points such that the angles PAQ and PBQ are constant; if P describe a conic through A and B, so will Q.*

Ex. 6. *$(PQR...)$ and $(P'Q'R'...)$ are two homographic ranges on the lines OA, OA'; if the parallelogram $POP'V$ be constructed, show that the locus of V is a conic.*

Viz. a conic through the points at infinity on OA and OA'.

Ex. 7. *PCP' and DCD' are fixed conjugate diameters of an ellipse. On CP and CD are taken X and Y such that*

$$PX \,.\, DY = 2\, CP \,.\, CD.$$

Show that DX and PY meet on the given ellipse.

For X and Y generate homographic ranges of which P and D are the vanishing points. To get the constant, take X at P'; then Y is at C.

9. *The locus of the meets of corresponding rays of two pencils whose corresponding angles are equal but measured in opposite directions is a rectangular hyperbola with the vertices of the pencils at the ends of a diameter.*

The pencils are clearly superposable and therefore homographic. Hence the locus is a conic through the vertices of the pencils.

Let O and V be the vertices and OB, VC corresponding rays meeting at A. Let AD bisect the angle OAV; and draw OE, VF parallel to AD. Then

$$EOA = OAD = DAC = FVA.$$

Let any other corresponding rays OQ, VR meet at P. Then by hypothesis $AOP = AVP$. Hence

$$EOP = EOA - AOP = FVA - AVP = FVP.$$

Hence to construct corresponding rays OQ and VR, we make the angles EOQ and FVR equal but in opposite directions. Now take each of these angles zero and we get the corresponding rays OE and VF which, being parallel, give a point on the locus at infinity. Hence one asymptote is parallel to OE. Again, make each of the angles a right

angle and we get the rays OG and VH which, being perpendicular to parallel lines, are parallel. Hence another asymptote is parallel to OG. Hence the asymptotes are perpendicular; and the conic is therefore a rectangular hyperbola.

Again, draw the tangents OT at O and VT' at V to the locus. Then OT corresponds to VO. Hence $EOT = FVO$. So $EOV = FVT'$. Hence

$$VOT = EOT - EOV = FVO - FVT' = OVT'.$$

Hence OT and VT' are parallel. Hence OV is a diameter.

Ex. 1. *The point of trisection of a given arc of a circle may be constructed as one of the meets of the arc with a rectangular hyperbola.*

Let AB be the arc and BT the tangent at B. Let C be the centre of the circle. Make the angle ACP equal to the angle TBP. Then if P is on the arc we have

$\angle BCP = 2\angle TBP = 2\angle ACP$. Hence arc $BP = 2$ arc AP.

If P is not on the arc, the locus of P is a rectangular hyperbola; and if Q be that meet of the circle and the rectangular hyperbola which lies between A and B, Q trisects the arc AB.

The other meets trisect the other arc AB and the arc supplementary to AB.

Ex. 2. *The locus of the points of contact of parallel tangents to a system of confocal conics is a rectangular hyperbola through the foci.*

***10.** Converse of Pappus's theorem. *If a point move so that its perpendicular distances $\alpha, \beta, \gamma, \delta$ from four fixed lines AB, BC, CD, DA are connected by the relation $\alpha \,.\, \gamma = k \,.\, \beta \,.\, \delta$, then the locus of P is a conic through $ABCD$.*

For $\dfrac{\alpha \,.\, \gamma}{\beta \,.\, \delta} \cdot \dfrac{AB \,.\, DC}{BC \,.\, AD}$ is constant. Hence, reasoning as in § 3, we see that $P(AC, BD)$ is constant.

Ex. 1. *Given two pairs of lines which are conjugate for a circle, the locus of the centre of the circle is a rectangular hyperbola.*

Let AB, CD be conjugate, and also BC, AD. Assume O to be a position of the centre. From O drop OP perpen-

dicular to DC to meet AB in P'. Then P' is the pole of CD, hence $OP . OP' = (\text{radius})^2$. So if OQ, perpendicular to AD, meet BC in Q', we have $OP . OP' = OQ . OQ'$. Also $OP = \gamma$, $OP' \propto a$, $OQ = \delta$, $OQ' \propto \beta$. Hence $a . \gamma \propto \beta . \delta$. Hence the locus of O is a conic through $ABCD$. Also the orthocentre of ADC gives $OP . OP' = OQ . OQ'$. Hence the conic is a r. h.

Ex. 2. *The locus of the foci of conics inscribed in a parallelogram is a r. h. circumscribing the parallelogram.*

Here $$a . \gamma = \beta . \delta.$$

11. *The projection of a conic is a conic.*

We have to prove that any projection of a conic can be placed in perspective with a circle. Now every projection of a conic is such that all the points on it subtend homographic pencils at two points on it; for this is true in the conic which was projected and is a projective property. Hence the projection is the locus of the meets of two homographic pencils and is therefore a conic. (See also VI. 7.)

12. *If two quadrangles have the same harmonic points, then their eight vertices lie on a conic; as a particular case, if any three of the vertices are collinear, the eight vertices lie on two lines.*

Let $ABCD$, $A'B'C'D'$ be the two given quadrangles, and UVW the common harmonic triangle.

If no three of the eight vertices lie on a line, we can draw a conic through any five, say A', B', C', D' and A. Then from the inscribed quadrangle $A'B'C'D'$ we see that UVW is a self-conjugate triangle for this conic. Also by hypothesis UVW is the harmonic triangle of the quadrangle $ABCD$. Hence (see figure of V. 9) B is such that $(WANB)$ is harmonic; hence B is on the conic, for A is on the conic, and W is the pole of UV; similarly C and D are on the conic.

Hence $ABCDA'B'C'D'$ lie on a conic.

This is true, however nearly three of the points lie on a line. Hence it is true when three of the points lie on a line. But as a curved conic cannot pass through three

points on a line, the conic must in this case degenerate into two lines. Hence in this case the eight vertices lie on two lines.

By reciprocation we deduce the theorem—*Two quadrilaterals which have the same harmonic triangle are such that the eight sides touch a conic* (*or pass through two points if three of the sides are concurrent*).

Ex. 1. *A conic can be drawn through the eight points of contact of two conics inscribed in the same quadrilateral.*

For the two sets of four points of contact have the same harmonic triangle, viz. that of the four common tangents.

Ex. 2. *The eight tangents at the four meets of any two conics touch the same conic.*

Reciprocate Ex. 1.

XI

1. The chords AB, CD of a conic are conjugate, ACB being a right angle. Through D is drawn the chord DP meeting AB at Q. Show that CA, CB are the bisectors of the angle PCQ.

2. If a variable circle divide a given arc of a given circle harmonically, it is orthogonal to the circle which passes through the ends of the given arc and is orthogonal to the given circle.

3. A fixed line DA meets a fixed conic at A, and EB touches at a fixed point B. A fixed point O is taken on the conic. Through A is drawn a variable line meeting the conic again at P and EB at Q. OP meets DA at U and OQ meets DA at V. Find the position of O when UV is of constant length.

4. All but one of the vertices of a polygon move on fixed lines and each side passes through a fixed point. Find the locus of the remaining vertex.

5. EF, FD, DE pass through the fixed points A, B, C. The centroid of DEF is fixed at G. AG is produced to H, so that $GH = 2 . AG$. Show that the locus of D is a conic through B, C, G, H.

6. A variable line PQ passes through a fixed point D and meets the fixed lines AB and AC at P and Q. Through P and Q are drawn PR and QR in given directions. Show

that the locus of R is a hyperbola with asymptotes in the given directions; and find where the locus meets AB and AC.

7. Show that the locus of the centre of the circle which circumscribes the triangle formed by two fixed tangents and any third tangent to a given circle is a hyperbola whose asymptotes are perpendicular to the fixed tangents.

8. Through a fixed point O is drawn a variable line to cut the sides BC, CA, AB of a triangle at P, Q, R; and on OP is taken the point X such that (PQ, RX) is a constant cross ratio. Show that the locus of X is a conic through O, A, B, C.

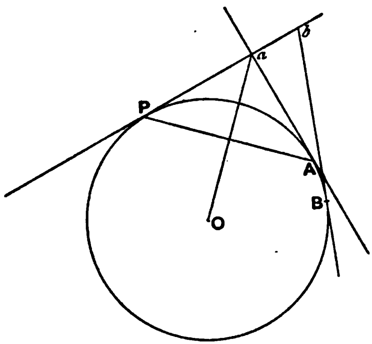
9. OP, OQ are the tangents from the fixed point O to one of a system of conics having the same foci S and H. Show that the conic $OPQSH$ passes through a fourth fixed point, viz. the intersection of the perpendiculars to OS at S and to OH at H.

CHAPTER XII

ANHARMONIC PROPERTIES OF TANGENTS OF A CONIC

1. *Four fixed tangents of a conic cut any variable fifth tangent of the conic in a constant cross ratio.*

Consider first the circle of which the conic is the projection. Let the fixed tangents of the conic be the projections of the tangents at $ABCD$ of the circle, and let the variable tangent of the conic be the projection of the variable tangent at P of the circle. Let the tangent at A cut the tangent at P in a, and so on.



Then if O be the centre of the circle, Oa is perpendicular to PA. Hence the pencils $O(abcd)$ and $P(ABCD)$ are superposable and therefore homographic. But $P(ABCD)$ is independent of the position of P on the circle. Hence $O(abcd)$, i.e. $(abcd)$, is independent of the position of the variable tangent of the circle. Hence the proposition is true for a circle; and being a projective theorem, it follows at once for the conic by projection.

The constant cross ratio (ab, cd) determined on a variable tangent by four fixed tangents is called a *cross ratio of the four tangents.*

Notice that the point where a tangent cuts itself is its point of contact; for as two tangents approach, their meet approaches the point of contact of each.

Similarly *any number of tangents of a conic determine on two other tangents of the conic two ranges which are homographic.*

Notice that we have in the above proof incidentally shown that *the range determined on any tangent of a conic by several other tangents of the conic is homographic with the pencil subtended at any point on the conic by the points of contact of the other tangents.*

Another proof of the first part comes from the fact that the range ab ... subtends at O in all its positions pencils which are superposable; a theorem which is tedious to prove on account of the number of cases to be considered.

Another proof is the following, by (1, 1) correspondence. Let a variable tangent to the conic cut the fixed tangents at A and B at the points P and Q. Then when P is given, PQ is given uniquely, being the tangent from P, other than PA; hence Q is given uniquely as the intersection of PQ with the tangent at B. So when Q is given, P is given uniquely. Hence P and Q are connected by a (1, 1) construction; which is clearly rational and hence P and Q generate homographic ranges.

Again take any point V on the conic and let PQ touch at R. Then VR gives R uniquely which gives PQ uniquely which cuts the tangent at A in a unique point P; so P gives R and therefore VR uniquely. Hence P and VR are connected by a (1, 1) construction; this is clearly rational and therefore P and VR generate a range and a pencil which are homographic.

Four tangents which meet any fifth tangent in a harmonic range are called *harmonic tangents*; and it follows by reciprocation that *if the tangents a, a' are harmonic with the tangents b, b', then a, a' and b, b' meet at conjugate points, and, conversely, the tangents from conjugate points are harmonic.*

Ex. 1. *A variable tangent of a conic meets at Q and Q' the tangents at the ends P, P' of a fixed diameter of the conic; show that $PQ \cdot P'Q' = CD^2$, CD being the semi-diameter conjugate to CP.*

For P and P' are the vanishing points of the homographic ranges determined by Q and Q' on the tangents at P and P'; for when Q is at P, Q' is at infinity and so for Q'. Hence $PQ \cdot P'Q'$ is constant. To get the constant in the ellipse,

take QQ' parallel to PP'. To get the constant in the hyperbola, take an asymptote as QQ'. Then in each case

$$PQ = P'Q' = CD.$$

Ex. 2. *A variable tangent to a conic meets the adjacent sides AB, BC of the parallelogram $ABCD$ circumscribed to the conic in P and Q; show that $AP \cdot CQ$ is constant.*

Ex. 3. *A variable tangent cuts the asymptotes of a hyperbola in T and T'; show that $CT \cdot CT'$ is constant, C being the centre.*

The tangent TT' rules homographic ranges on the tangents CT, CT'. Also when T is at C, T' is at infinity, and when T' is at C, T is at infinity; hence I and J' are at C.

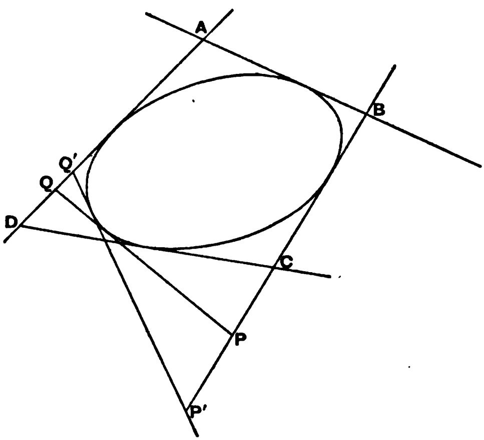
Ex. 4. *Deduce the equation of a hyperbola referred to its asymptotes, viz. $xy =$ constant.*

Since $PT = PT'$, $x = \frac{1}{2}CT$, $y = \frac{1}{2}CT'$.

Ex. 5. *B and C are the points of contact of tangents from A to a conic. A variable tangent meets AB in P and AC in Q. Show that the locus of $(BQ; CP)$ is a conic touching the given conic at B and C.*

Ex. 6. *If AA', BB' be pairs of harmonic points on a conic, show that the four tangents at $ABA'B'$ cut any fifth tangent in a harmonic range.*

***2.** *If AB, BC, CD, DA touch a conic, and p, q, r, s be the*



perpendiculars from A, B, C, D on a variable tangent of the conic, then $p \cdot r = k \cdot q \cdot s$.

Let two variable tangents cut BC in P, P' and AD in Q, Q'.

$$\text{Then}\quad (BC,\ PP') = (AD,\ QQ').$$

$$\therefore\ \frac{BP}{PC} \cdot \frac{P'C}{BP'} = \frac{AQ}{QD} \cdot \frac{Q'D}{AQ'};$$

$$\therefore\ BP.QD \div PC.AQ \quad \text{is constant.}$$

$$\text{But}\ \frac{BP}{PC} = \frac{q}{r} \ \text{and}\ \frac{AQ}{QD} = \frac{p}{s},$$

$$\therefore\ p.r \div q.s \quad \text{is constant.}$$

Ex. 1. *Extend the theorem to a 2n-sided circumscribed polygon.*

Ex. 2. *Deduce a theorem concerning an n-sided circumscribed polygon.*

Ex. 3. *If the conic be a circle, show that $p.r \div q.s$ is equal to $OA.OC \div OB.OD$, O being the centre.*

For $\sin AOQ = \sin BOP$.

Ex. 4. *If the conic be a parabola, then $p.r = q.s$.*

For taking the line at infinity as tangent,

$$k = p'.r' \div q'.s' = 1.$$

Ex. 5. *Show that for any conic the k of $p.r = k.q.s$ is the cross ratio of the four tangents divided by the cross ratio of the pencil formed by four lines drawn parallel to them through any vertex.*

Let PQ meet AB in M and CD in N. Then

$$MQ \div \sin MAQ = AQ \div \sin AMQ;\ \text{and so on.}$$

Hence the ratio of cross ratios corresponding to $(MN,\ QP)$ is

$$AQ.PC \div QD.BP = p.r \div q.s.$$

Ex. 6. *The sides BC, CA, AB of a triangle touch a conic at P, Q, R; show that if t be any tangent*

$$\text{(i)}\ (P,\ t).(A,\ t) \propto (B,\ t).(C,\ t);$$
$$\text{(ii)}\ (R,\ t).(Q,\ t) \propto (A,\ t)^2.$$

***3.** *Deduce, from the theorem $\alpha.\gamma = k.\beta.\delta$ of XI. 3, the theorem $p.r = k.q.s$ by reciprocation.*

Call the sides of the inscribed figure in XI. 3 a, b, c, d; and let the reciprocals of a, b, c, d be the points A, B, C, D of a four-sided figure circumscribing a conic; then p, the reciprocal of P, touches this conic.

The given theorem $\alpha . \gamma = k . \beta . \delta$ asserts that

$$(P, a) . (P, c) \div (P, b) . (P, d)$$

is constant.

But by Salmon's theorem $OP/(P, a) = OA/(A, p)$, and so on.

Hence, dividing by OP^2, we see that

$$\frac{(A, p)}{OA} \cdot \frac{(C, p)}{OC} \div \frac{(B, p)}{OB} \cdot \frac{(D, p)}{OD}$$

is constant.

Now O is a fixed point, hence

$$(A, p) . (C, p) \div (B, p) . (D, p)$$

is constant, i. e. $p . r \div q . s$ is constant.

Ex. 1. *Given any fixed point O and any conic, two lines s and h can be found such that $OP^2 \div (P, s) . (P, h)$ is constant, P being a variable point on the conic.*

Viz. the lines corresponding to the foci of a reciprocal of the conic for O.

Ex. 2. *AA', BB', CC' are the three pairs of opposite vertices of a quadrilateral circumscribed to a parabola whose focus is S; show that $SA . SA' = SB . SB' = SC . SC'$.*

Take the four-sided figure whose vertices are $AB'A'B$. Then $p . r = q . s$. Hence in the reciprocal circle we have

$$SA . SA' . \alpha . \gamma = SB . SB' . \beta . \delta.$$

But $k = 1$ in the circle. Hence

$$SA . SA' = SB . SB', = SC . SC' \text{ similarly.}$$

Ex. 3. *If the tangents at $ABCD$... to a circle meet in LMN..., then, t being any tangent and O the centre of the circle,* $\Pi (A, t) : \Pi (L, t) :: \Pi\, OA : \Pi\, OL$, Π *denoting a product.*

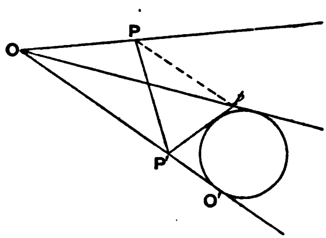
For $\Pi (T, a) = \Pi (T, l)$ in a circle.

4. *The lines joining corresponding points of two homographic ranges which are on different axes and not in perspective touch a conic which touches the axes.*

Let the ranges be $(PQR \ldots)$ and $(P'Q'R' \ldots)$ on the axes OP and OP'. Since they are not in perspective, the point which corresponds to O in the range $(P'Q'R' \ldots)$ will be some point O' not coinciding with O. Draw any circle touching OP' at O', and from O and P' draw the second tangents to this circle, meeting in p.

Then the range (P) = range (P') by hypothesis = range (p) from the circle. Hence the ranges (P) and (p) are homographic. Also when P' coincides with O', both P and p coincide with O. Hence the ranges are in perspective.

Now rotate the figure of the circle out of the original plane about the axis OO'. Then the ranges (P) and (p) are still in perspective. Hence all the lines Pp, Qq, Rr, ... meet in a point, say V. Hence, taking V as vertex of projection, p projects into P, and therefore the line $P'p$ into the line $P'P$. Hence, since $P'p$ in all positions touches a circle, $P'P$ in all positions touches the projection of a circle, i.e. a conic. Also, since the circle touches Op and OP', the conic touches the projections of these lines, viz. OP, and OP'.



To find the point of contact of OP with the envelope, we make PP' almost coincide with OP so that P' almost coincides with O. Hence the required point P is the point on OP corresponding to O on OP'. So the point of contact of OP' is the point on OP' corresponding to O on OP.

Notice that if the ranges be in perspective the envelope of PP' degenerates into the centre of perspective and the meet of the axes of the ranges.

5. *One, and only one, conic can be drawn touching five given lines.*

The envelope of a line which is cut by four given lines in a given cross ratio is a conic touching the given lines.

These propositions can be proved like the reciprocal propositions in XI. 6 or they may be deduced from these by reciprocation.

6. *Every two conics have four common tangents.*

Two conics cannot have more than four common tangents; for if they had five, they would coincide. Also we see that

two equal ellipses laid across one another have four common tangents. Hence we conclude that if two conics have not apparently four common tangents, some of the tangents are imaginary, or coincident.

Touching four given lines can be drawn an infinite number of conics.

For we can draw a conic touching the four given lines and any fifth line.

All the conics which touch four given lines have a common self-conjugate triangle, viz. the harmonic triangle of the quadrilateral formed by the common tangents.

Ex. 1. *Given two homographic ranges ABC ... and $A'B'C'$ ... on different lines; show that two points can be found at each of which the segments AA', BB', CC', ... subtend the same angle.*

Viz. the foci of the conic touching the lines and AA', ...

Ex. 2. *The vertices A, B, C of a triangle lie on the fixed lines MN, NL, LM, and the sides BA, AC pass through the fixed points W and V; show that the envelope of BC is a conic touching the five lines LM, LN, VW, NV, MW.*

Ex. 3. *From the variable point O situated on a fixed line are drawn the lines OA, OB, OC to the fixed points ABC, meeting BC, CA, AB in X, Y, Z; BC, YZ meet in X', CA, ZX meet in Y', and AB, XY meet in Z'. Show that the line $X'Y'Z'$ envelopes a conic which touches each side of the triangle at the fourth harmonic of the fixed line for the side.*

By a previous example $X'Y'Z'$ are collinear. Also

$$(O) = A\,(O) = (X) = (X') \text{ since } (BC, XX') \text{ is harmonic}$$
$$= (Y') \text{ similarly.}$$

Hence $X'Y'$ envelopes a conic touching CB and CA and similarly AB. To find the point of contact X' of CB, we must take Y' at C; then Y is at C, hence O is on BC, i.e. O is at the intersection P, say, of the locus of O with CB. Hence X is also at P. Hence X' is known, viz. the fourth harmonic of P for C and B.

Ex. 4. *The vertices BC of a triangle lie on given lines and the vertex A lies on a conic on which also lie fixed points VW through which the sides CA, AB pass. Show that the envelope of BC is a conic touching the given lines.*

Ex. 5. *The side BC of a triangle touches a conic, and the vertices B and C move on fixed tangents of this conic, whilst the*

sides AB, AC pass through fixed points; show that the locus of A is a conic through the fixed points.

Ex. 6. *Given in position two pairs of conjugate diameters of a conic and a tangent, construct the conic.*

Construct the parallel tangent (which is equidistant from the centre). Let these tangents cut a pair of conjugate diameters in LL' and MM'. Then LM and $L'M'$ also touch the conic. Proceeding similarly with the other pair, we have six tangents.

Ex. 7. *Given in position a pair of conjugate diameters and two tangents, construct the conic.*

7. *Any number of tangents of a parabola determine on two other tangents of the parabola two ranges which are similar.*

Let the two ranges be $(PQR\ldots)$ and $(P'Q'R'\ldots)$. Let Ω and Ω' be the two points at infinity upon the lines PQ and $P'Q'$. Then since the line at infinity touches the parabola, the line $\Omega\Omega'$ is a tangent. Hence the two ranges $(\Omega PQR\ldots)$ and $(\Omega' P'Q'R'\ldots)$ are homographic; also the points at infinity $\Omega\Omega'$ correspond. Hence the ranges are similar.

Conversely, *the lines joining corresponding points of two similar ranges which are on different axes and not in perspective touch a parabola which touches the axes.*

For if the ranges $(PQR\ldots)$ and $(P'Q'R'\ldots)$ are similar, the ranges $(\Omega PQR\ldots)$ and $(\Omega' P'Q'R'\ldots)$ are homographic. Hence the lines $\Omega\Omega'$, PP', QQ', ... all touch a conic which touches PQ and $P'Q'$. And this conic is a parabola since $\Omega\Omega'$ touches it.

Ex. 1. *One and only one parabola can be drawn touching four given lines.*

Ex. 2. *The envelope of a line which is cut by three given lines in a constant ratio is a parabola.*

Ex. 3. *Two parabolas have generally three finite common tangents.*

Ex. 4. *Touching three given lines can be drawn an infinite number of parabolas.*

Ex. 5. *TP, TP' touch a parabola at P and P', and cut a third tangent in Q, Q'; show that $QP : TP :: TQ' : TP'$.*

For $(QT, P\Omega) = (Q'P', T\Omega')$, considering the ranges determined on the two tangents TP, TP' by the four tangents QQ', TP', PT, $\Omega\Omega'$.

Ex. 6. *Also if QQ' touch at R, then $PQ/QT = QR/RQ'$.*

Ex. 7. *The envelope of the axes of conics which touch two given lines at given points is a parabola.*

Let TP, TP' be the fixed tangents. Then, if the normals at P and P' meet one axis at G, G' and the other at g, g', we know that

$$PG = \frac{b}{a}CD,\ PG' = \frac{b}{a}CD',\ Pg = \frac{a}{b}CD,\ Pg' = \frac{a}{b}CD'.$$

Hence $PG : PG' :: Pg : Pg' :: CD : CD'$. Now CD, CD' are parallel to TP, TP'. Hence $CD : CD' = TP : TP'$ is constant. Hence the axes GG', gg' envelope a parabola.

Ex. 8. *The normals at the points P and P' on a conic, the chord PP' and the axes of the conic touch a parabola.*

Ex. 9. *The ends PQ of a segment move on fixed lines, and the orthogonal projection of PQ on a fixed line is of constant length; show that the envelope of PQ is a parabola whose axis is in the direction of the projecting lines.*

Let pq be the projection of PQ. Then range (P) is similar to range (p), which is equal to range (q), which is similar to range (Q). Also when pq approaches infinity, PQ approaches being perpendicular to pq.

Ex. 10. *From points P on one line are drawn perpendiculars PQ, PR on two other lines; show that QR touches a parabola.*

Ex. 11. *If through any point parallels be drawn to the tangents of a parabola, a pencil is constructed homographic with the range determined by the tangents on any tangent.*

For the rays are the lines joining the point to the points where the line at infinity cuts the tangents.

Ex. 12. *If through points of a range on a given line there be drawn lines parallel to the corresponding rays of a pencil, which is homographic with the given range, these lines will touch a parabola.*

XII

1. If the connectors of the ends P and P' of a diameter of a conic to a point on the conic cut the tangents at P' and P at Q' and Q, show that $PQ \,.\, P'Q' = 4\,CD^2$.

2. On a fixed tangent of a conic are taken two fixed points A, B and also two variable points Q, R such that (AB, QR) is harmonic. Show that the locus of the intersection of the other tangents from Q and R is the connector of the points of contact of the other tangents from A and B.

3. The lines AB, BC, CD, DA touch a conic. One tangent meets AB, CD at M, N and another tangent meets AD, BC at P, Q. Show that

$$AM \,.\, BQ \,.\, CN \,.\, DP = AP \,.\, BM \,.\, CQ \,.\, DN.$$

4. $ABC\ldots$ and $A'B'C'\ldots$ are homographic ranges on different lines. Show that two points can be found at which AA', BB', CC', ... all subtend angles having the same bisectors.

5. All but one of the sides of a polygon pass through fixed points and each vertex moves on a fixed line. Find the envelope of the remaining side.

6. If $e\,(ab, cd)$ mean the cross ratio determined on the line e by the lines a, b, c, d, show that

$$e\,(ab, cd) \,.\, c\,(ab, de) \,.\, d\,(ab, ec) = 1,$$

a, b, c, d, e being any five lines.

7. Through the fixed points A, B is drawn a variable circle meeting fixed lines through A at P, Q. Find the envelope of PQ.

8. Determine (as a common tangent of two parabolas) a line which shall meet given lines AA', BB', CC' in points P, Q, R such that $AP = BQ = CR$.

9. If all the tangents of a parabola be turned through the same angle in the same direction about the points in which they meet a fixed tangent, they will still touch a parabola.

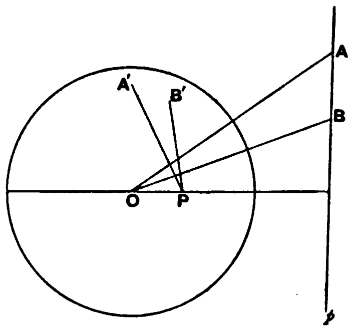
10. Given any four lines, a point can be found such that the feet of the perpendiculars from the point on the lines are collinear.

CHAPTER XIII

POLES AND POLARS. RECIPROCATION

1. *A* RANGE *formed by any number of points on a given line is homographic with the pencil formed by the polars of these points for a conic.*

Consider the circle of which the conic is the projection. Let A, B, ... on the line p be the points in the figure of the circle which project into the points on the given line in the figure of the conic.



Now since $A, B, \ldots$ lie on p, the polars PA', PB', ... all pass through P, the pole of p. Also PA' is perpendicular to OA, O being the centre of the circle. Hence the pencil $P(A'B'\ldots)$ is superposable to and therefore homographic with the pencil $O(AB\ldots)$, and is therefore homographic with the range $(AB\ldots)$. Hence the proposition is true for a circle; and being a projective theorem, it is true for the conic by projection.

Taking the base conic as the given conic, the theorem becomes:—

The reciprocal of a range of points is a pencil of lines which is homographic with the given range.

Another proof is by (1, 1) correspondence, thus:—Take a variable point P on a fixed line l. The polar of P is a variable line p through the fixed point L, the pole of l; hence as P generates a range, p generates a pencil. Also when P is given, p is known uniquely; and when p is given,

P is known uniquely; also the construction is rational. Hence the range and the pencil are homographic.

Ex. 1. *Through a fixed point O is drawn a variable line cutting a fixed line in Q' and a fixed conic in PP'. If (PP', QQ') be harmonic, show that the locus of Q is a conic passing through O, through the pole of the fixed line, through the meets of this line with the conic, and through the feet of the tangents from O.*

For $O(Q) = (Q') = V(Q)$, V being the pole of the locus of Q'.

Ex. 2. *Obtain the reciprocal theorem to that of Example* 1.

Ex. 3. *If P, P' move on fixed lines l, l'; then, if P, P' are conjugate for a conic, they generate homographic ranges. Also if p, p' pass through fixed points; then if p, p' are conjugate for a conic, they generate homographic pencils.*

For, if L be the pole of l, then

$$(P) \text{ of poles} = L(P') \text{ of polars} = (P').$$

For the second part, reciprocate.

Ex. 4. *Show that in Ex.* 3, *PP' envelopes a conic which touches l and l' and also the four tangents of the conic at its intersections with l and l'. Show also that if l and l' are conjugate lines, the envelope degenerates into the points L and L'; and if the intersection of l, l' is on the conic, into this point and another point.*

Ex. 5. *Two vertices of a triangle self-conjugate for a given conic move on fixed lines; show that the locus of the third vertex is a conic passing through the intersections of the given lines with the given conic and through the poles of the given lines for the given conic.*

Ex. 6. *AA' are a pair of opposite vertices of a quadrilateral whose sides touch a conic at L, M, N, R. Through A and A' are drawn conjugate lines meeting in P. Show that the locus of P is the conic $AA'LMNR$.*

Ex. 7. *Through a fixed point O is drawn a variable line, and PY is the perpendicular on this line from its pole P for a fixed conic; show that PY envelopes a parabola, which touches the polar of O, and also touches the tangents at the feet of the normals from O.*

Let PY cut the line at infinity in Q. Through any point V draw Vq parallel to PY; then Vq passes through Q. Hence $(Q_1Q_2\ldots) = V(q_1q_2\ldots) = O(Y_1Y_2\ldots)$ [corresponding rays being perpendicular] $= (P_1P_2\ldots)$. Hence PQ, i. e. PY,

envelopes a conic touching P_1P_2 and Q_1Q_2, i. e. the polar of O and the line at infinity. This parabola touches the tangent at R, a foot of a normal from O; for if OY be OR, then PY is the tangent at R.

Ex. 8. *If instead of being perpendicular to the variable line, PY make a given angle with it; show that PY envelopes a parabola, which touches the polar of O, and also touches the tangents at the points where the tangents make the above angle with the radii from O.*

Ex. 9. *Through points PQ... on the line l are drawn the lines PP', QQ', ... parallel to the polars of P, Q, ... for a conic; show that PP', QQ', ... touch a parabola which touches l.*

2. *The reciprocal of a conic for a conic is a conic.*

We may define the original conic as the locus of a point P such that $P(ABCD) = E(ABCD)$, where A, B, C, D, E are fixed points on the conic. Let the reciprocals of the points A, B, C, D, E, P be the lines a, b, c, d, e, p. Now the reciprocal of the pencil $P(ABCD)$ is the range of points determined on the line p by the lines a, b, c, d. Hence this range is homographic with $P(ABCD)$. So the range of points determined on e by a, b, c, d is homographic with $E(ABCD)$, i. e. with $P(ABCD)$, i.e. with the range of points determined on p by a, b, c, d. Hence the reciprocal of the given conic, viz. the envelope of p, the reciprocal of P, is the envelope of a line which cuts four given lines a, b, c, d in a constant cross ratio. Hence the reciprocal is a conic touching a, b, c, d, e.

3. *The reciprocal of a pole and polar for a conic is a polar and pole for the reciprocal conic.*

Let P be the pole and e its polar. Through P draw any line r cutting e in P' and the conic in Q, Q'. Then (PP', QQ') is harmonic. Let the reciprocals of P, e, r, P', Q, Q' be p, E, R, p', q, q'. Then on a fixed line p is taken a variable point R, and from R are drawn the tangents q, q' to the reciprocal conic, and the line p' is taken such that (pp', qq') is harmonic. We are given that p' always passes through E, and we have to prove that E is the pole of p. But this

is obvious, for p and p' are conjugate in all positions of p', since $(pp', qq') = -1$. Hence p' always passes through the pole of p, i.e. E is the pole of p.

Ex. 1. *The reciprocal of a triangle self-conjugate for a conic is a triangle self-conjugate for the reciprocal conic.*

Ex. 2. *A triangle self-conjugate for the base conic reciprocates into itself.*

Ex. 3. *A conic, its reciprocal, and the base conic have a common self-conjugate triangle.*

Viz. the common self-conjugate triangle of the given conic and the base conic.

4. *Given any two conics, a base conic can be found for which they are reciprocal.*

Let UVW be the common self-conjugate triangle of the two conics. Project the line VW to infinity and the angle VUW into a right angle. Then U becomes the centre of both conics since U is now the pole of the straight line at infinity for each conic. Also VU and WU are now the principal axes of both conics, for VU and WU are perpendicular conjugate lines at the centre. Let UV cut one conic c_1 at A, A' and the other conic c_2 at L, L' and let UW cut c_1 at B, B' and c_2 at M, M'. Take a conic Γ with principal semi-axes UX, UY along UV, UW such that

$$UX^2 = UA \,.\, UL \text{ and } UY^2 = UB \,.\, UM.$$

Then the two conics c_1 and c_2 are reciprocal for Γ.

For consider the polar of A for Γ. Let it cut UV at A_1. Then $UA \,.\, UA_1 = UX^2$. Hence A_1 coincides with L. Hence by symmetry the polar of A is the perpendicular to UV through L, i.e. is the tangent to c_2 at L. So the polar of L for Γ is the tangent at A to c_1. Hence A and the tangent at A to c_1 reciprocate into the tangent at L to c_2 and L, i.e. the reciprocal of c_1 touches c_2 at L. So it touches c_2 at L', M and M'. Hence c_2 has eight points in common with the reciprocal of c_1 for Γ and therefore coincides with the reciprocal. Hence c_1 and c_2 are reciprocal for Γ in the projected figure.

But the reciprocal property is not changed by projection; for it depends entirely on properties of poles and polars which are unchanged by projection. Hence in the original figure a conic can be found for which the original conics are reciprocal.

There are many cases of failure in the above proof. If the failure arises from certain points being imaginary, we appeal to the principle of continuity. If the cause of failure is that certain intersections of the conics coincide, then as the proposition is true however near the points are, we may assume it is true when they coincide.

For an exhaustive study of the problem of this article, the reader is referred to a paper by the author in the *Proceedings of the London Mathematical Society*, Vol. XXVI.

Notice that in any case *a conic is its own reciprocal*, viz. with respect to itself.

Ex. *The cross ratio of the four common points of two conics for one of the conics is equal to the cross ratio of the four common tangents for the other conic.*

5. *Reciprocate—a segment divided in a given ratio.*

Let AC be divided in B. Let l be the line AB and i the line at infinity, and let Ω be the meet of l and i. The reciprocals of the points $ABC\Omega$ on the line l are the lines $abc\omega$ through the point L. Also the reciprocal of i is the centre O of the base conic. Hence $AB \div BC = -(AC, B\Omega)$ of the given range of points $= -(ac, b\omega)$ of the reciprocal pencil, where ω is the join of L to O.

As a particular case *the middle point of a segment AC reciprocates into the fourth harmonic for a and c of the join of ac to the centre of the base conic.*

Ex. *Reciprocate the theorem—*

'*The locus of the centres of conics inscribed in a given quadrilateral is a line which bisects each of the three diagonals.*'

XIII

1. AP, AQ which are harmonic with two fixed lines through A meet a conic at P, Q. Show that the envelope of

PQ is a conic touching the fixed lines at the points on the polar of A, and touching the tangents of the conic at its intersections with the fixed lines.

2. Through a fixed point O is drawn a variable line OY, and PY is drawn making a fixed angle with OY through the pole P of OY. If this angle is the angle between CO and the polar of O, show that the locus of Y is a circle.

3. If through every point of a line there be drawn the chord of a given conic which is bisected at the point, show that the envelope of the chord is a parabola which touches the line.

4. The reciprocals of the four points A, B, P, Q are the four lines a, b, p, q. Show that

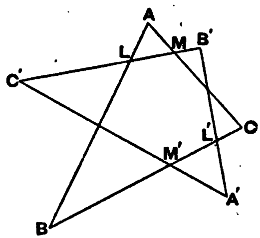
$$\frac{(P, a)}{(P, b)} \div \frac{(A, p)}{(B, p)} = \frac{(Q, a)}{(Q, b)} \div \frac{(A, q)}{(B, q)}.$$

CHAPTER XIV

PROPERTIES OF TWO TRIANGLES

1. *If the vertices of two triangles lie on a conic, the sides touch a conic; and conversely.*

Let the vertices ABC, $A'B'C'$ of the two triangles lie on a conic. Let AB, AC meet $B'C'$ in L, M; let $A'B'$, $A'C'$ meet BC in L', M'. Then



$$(C'LMB') = A\,(C'BCB')$$
$$= A'\,(C'BCB') = (M'BCL').$$

Hence the six lines $C'M'$, LB, MC, $B'L'$, $B'C'$, BC touch a conic; i.e. $C'A'$, AB, AC, $B'A'$, $B'C'$, BC touch a conic; i.e. the sides of the triangles touch a conic.

Let the sides touch a conic. Then

$$A\,(C'BCB') = (C'LMB') = (M'BCL') = A'\,(C'BCB').$$

Hence the six points C', B, C, B', A, A' lie on a conic; i.e. the vertices lie on a conic.

Ex. 1. *If two conics be such that one triangle can be drawn which is circumscribed to one conic and inscribed in the other, then an infinite number of such triangles can be drawn.*

For suppose ABC to be circumscribed to β and inscribed in γ. Draw *any* tangent to β cutting γ in B' and C'. From B' and C' draw the other tangents to β meeting in A'. Then, since ABC, $A'B'C'$ are circumscribed to β, the vertices $ABCA'B'C'$ lie on one conic; hence A' lies on γ. Hence $A'B'C'$ satisfies the required conditions.

Ex. 2. *If BC be the points of contact of tangents from A, and $B'C'$ be the points of contact of tangents from A' to a conic; show that the triangles ABC, $A'B'C'$ are inscriptible in a conic, and circumscriptible to a conic.*

Let AB, AC cut $B'C'$ in L, M; let $A'B'$, $A'C'$ cut BC in L', M'. Then $(LB'C'M)$ of poles $= A'(BL'M'C)$ of polars.

Hence $(LB'C'M) = (BL'M'C)$. Hence the triangles are circumscriptible, and therefore inscriptible.

See also XXIX. 5, Ex. 3.

Ex. 3. *If O be the centre of the conic circumscribing ABC, $A'B'C'$ (of Ex. 2), and if BC and $B'C'$ meet in D, show that DO bisects AA'.*

For D is the pole of AA' for the given conic and therefore for the new conic from the quadrangle $BB'C'C$.

Ex. 4. *A conic is drawn through a fixed point A and through the points of contact B, C of tangents from A to a circle, so as to touch the circle at a variable point P. Show that the curvatures of all the conics at the points P are equal.*

In Ex. 2 let $A'B'C'$ coincide in P. Then the circle of curvature of the conic at P is the circum-circle of $A'B'C'$, whose radius is one-half of that of the given circle.

2. *If two triangles be self-conjugate for a conic, the six vertices lie on a conic, and the six sides touch a conic; conversely, if the six vertices lie on a conic, or if the six sides touch a conic, the triangles are self-conjugate for a conic.*

In the figure of § 1, let ABC, $A'B'C'$ be self-conjugate for a conic. Then the polar of C' is $A'B'$, the polar of L where $B'C'$ and AB meet is $A'C$, the polar of M where $B'C'$ and AC meet is $A'B$, and the polar of B' is $A'C'$. Hence

$$(C'LMB') \text{ of poles} = A'(B'CBC') \text{ of polars}$$
$$= (L'CBM') = (M'BCL').$$

Hence the six sides $C'M'$, LB, MC, $B'L'$, $B'C'$, BC touch a conic; and hence the six vertices lie on a conic.

If the two triangles are inscriptible in a conic γ, describe by XXV. 12 a conic α such that ABC is self-conjugate for α, and that A' is the pole of $B'C'$ for α. Let the polar of B' for α cut $B'C'$ in C''. Then ABC and $A'B'C''$ are self-conjugate for α; hence $ABCA'B'C''$ lie on a conic. But this conic is γ, for the points $ABCA'B'$ lie on both conics. Hence $B'C'$ cuts γ in three points unless C' and C'' coincide. Hence C' and C'' coincide. Hence ABC, $A'B'C'$ are self-conjugate for a conic, viz. for the conic α.

If the two triangles are circumscribed to a conic, they are also inscribed in a conic, and the above proof applies.

Ex. 1. *If two triangles be self-conjugate for a conic α, then a conic β drawn to touch five of the sides will touch the sixth also, and a conic γ drawn to pass through five of the vertices will pass through the sixth also; and γ and β are reciprocal for α.*

For the reciprocals of five points on γ touch β.

Ex. 2. *Through the centre of a conic and the vertices of a triangle self-conjugate for the conic can be drawn a hyperbola with its asymptotes parallel to any pair of conjugate diameters of the conic.*

Let ABC be the triangle and $OP\Omega$ and $OD\Omega'$ the conjugate diameters. Then a hyperbola passes through ABC, $O\Omega\Omega'$, since $O\Omega\Omega'$ is also self-conjugate for the conic.

Ex. 3. *If two conics be such that one triangle can be circumscribed to one conic which is self-conjugate for the other conic, then an infinite number of such triangles can be drawn.*

Let ABC be the given triangle touching conic β and self-conjugate for conic α. Take any tangent $B'C'$ of β, and take its pole A' for α; draw from A' one tangent $A'B'$ to β, and take C', the pole of $A'B'$ for α. Then, since ABC, $A'B'C'$ are self-conjugate for α, the sides touch a conic. But five sides touch β; hence the sixth side $C'A'$ touches β. Hence $A'B'C'$ satisfies the required conditions.

Ex. 4. *If two conics be such that one triangle can be inscribed in one conic which is self-conjugate for the other conic, then an infinite number of such triangles can be drawn.*

Reciprocate Ex. 3.

Ex. 5. *Two conics β and α are such that triangles can be circumscribed to β which are self-conjugate for α. Show that the harmonic locus of the conics is the reciprocal of β for α.*

The *harmonic locus* is the locus of a point P from which the pairs of tangents to the conics are harmonic. Let the tangents from P to β cut the polar of P for α at Q, R. Then, since the lines PQ, PR are (by hypothesis) harmonic with the tangents from P to α, they are conjugate for α. But QR is the polar of P for α; hence PQR is self-conjugate for α. And two of its sides touch β. Hence QR also touches β. Hence P, its pole for α, is on the reciprocal of β for α.

Ex. 6. *If two conics are such that triangles can be inscribed in one which are self-conjugate for the other, find the harmonic envelope of the conics.*

Reciprocate Ex. 5.

Ex. 7. *If Q and R be the points of contact of the tangents from P to any conic α, and any conic γ be drawn to pass through P and to touch QR at Q, then triangles can be inscribed in γ which are self-conjugate for α.*

For PQQ is such a triangle, QQ being QR.

Ex. 8. *If Q and R be the points of contact of the tangents from P to any conic α, and any conic β be drawn to touch PQ at P and to touch QR, then triangles can be circumscribed to β which are self-conjugate for α.*

For PQQ is such a triangle, QQ being QR.

Ex. 9. *If triangles can be circumscribed to β which are self-conjugate for α, then triangles can be inscribed in α which are self-conjugate for β; and conversely.*

For we can reciprocate α into β.

Ex. 10. *The triangle ABC is inscribed in the conic α, and the triangle DEF is self-conjugate for α. Show that a conic β can be found such that DEF is circumscribed to β and ABC is self-conjugate for β.*

Viz. that conic inscribed in DEF for which A is the pole of BC.

Ex. 11. *The conic α is drawn touching the lines PQ, PR at Q, R; the conic β is drawn touching the lines QP, QR at P, R; show that* (i) *triangles can be inscribed in α which are self-conjugate for β,* (ii) *triangles can be inscribed in β which are self-conjugate for α,* (iii) *triangles can be circumscribed to α which are self-conjugate for β,* (iv) *triangles can be circumscribed to β which are self-conjugate for α,* (v) *triangles can be inscribed in α and circumscribed to β,* (vi) *triangles can be inscribed in β and circumscribed to α.*

On RP and RQ take L, L' consecutive to R; on PR, QR take M, M' consecutive to P, Q; on QP, PQ take N, N' consecutive to Q, P. Then consider the triangles (i) QRL, (ii) PRL', (iii) QPM, (iv) PQM', (v) RQN, (vi) RPN'.

Such conics may be said to be *manifoldly related.*

Ex. 12. *If a triangle can be drawn inscribed in α and circumscribed to β and also a triangle self-conjugate for α and circumscribed to β, then the conics α and β are manifoldly related.*

At R, one of the meets of α and β, draw RQ touching β and meeting α again in Q; draw the tangent at Q, and on it take N consecutive to Q. Then by the first datum QN touches β, at P say. Then by the second datum QR is the polar of P for α, i.e. PR touches α at R.

Similarly many other converses of Ex. 11 can be proved.

Ex. 13. *The centre of a circle touching the sides of a triangle self-conjugate for a rectangular hyperbola is on the r. h.*

For, since a triangle circumscribes the circle and is self-conjugate for the r. h., triangles can be inscribed in the r. h. which are self-conjugate for the circle. Now one triangle self-conjugate for the circle is $O\Omega\Omega'$, and two of its vertices $\Omega\Omega'$ lie (at infinity) on the r. h.; hence O, the centre of the circle, lies on the r. h.

Ex. 14. *Given a triangle self-conjugate for a r. h., we know four points on the r. h.*

Viz. the centres of the touching circles.

Ex. 15. *Given a self-conjugate triangle of a conic and a point on the director, show that four tangents are known, viz. the directrices of the four conics which can be drawn to circumscribe the triangle and to have the point as corresponding focus.*

Reciprocate for the point.

Ex. 16. *The necessary and sufficient condition that triangles can be circumscribed to a circle which are self-conjugate for a r. h. is that the centre of the circle shall be on the r. h.*

Ex. 17. *An instance of Ex. 11 is a rectangular hyperbola which passes through the vertices of a triangle and also through the centre of a circle touching the sides.*

This follows from Ex. 12 and Ex. 16.

Ex. 18. *If two conics β and γ be so situated that one triangle can be circumscribed to β so as to be inscribed in γ, then an infinite number of such triangles can be drawn, and all of these will be self-conjugate for a third conic α; also the two conics β and γ are reciprocal for α.*

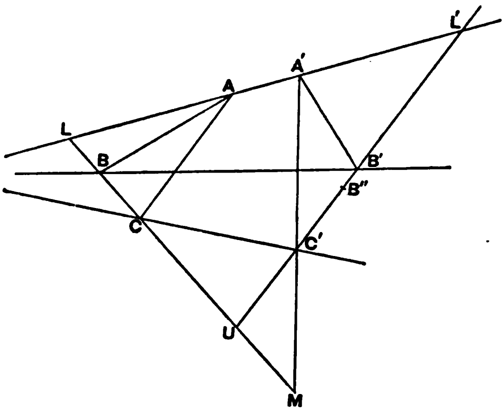
The first part has been proved. To prove the third part, notice that ABC, $A'B'C'$ are self-conjugate for a conic α. Define γ by $ABCA'B'$; then since the polars of these points for α, viz. BC, CA, AB, $B'C'$, $C'A'$ touch β, it follows that β is the reciprocal of γ for α.

Again, take any point A'' on γ, and let B'' be one of the points in which the polar of A'' for α (which touches β) cuts γ. Let the polar of B'' for α (which touches β and passes through A'') cut the polar of A'' in C''. Then the triangle $A''B''C''$ is self-conjugate for α. Hence, since two sides touch β and two vertices are on γ, it is circumscribed to β and inscribed in γ.

3. The *two triangles* ABC, $A'B'C'$ are said to be *reciprocal for a conic* if A be the pole of $B'C'$, B of $C'A'$, C of $A'B'$, A' of BC, B' of CA and C' of AB for the conic.

Two triangles which are reciprocal for a conic are homological; and conversely, if two triangles be homological they are reciprocal for a conic.

Let the triangles ABC, $A'B'C'$ be reciprocal for a conic; then they are homological. For let BC and $B'C'$ meet in U,



and let AA' meet BC in L and $B'C'$ in L'. Then we want to prove that AA', BB', CC' meet in a point; which is true if the ranges $(LBCU)$ and $(L'B'C'U)$ are homographic. But the polar of B is $A'C'$, the polar of C is $A'B'$, the polar of U where BC and $B'C'$ meet is $A'A$, the polar of L where BC and $A'A$ meet is $A'U$. Hence $(LBCU)$ of poles $= A'(UC'B'L')$. Hence $(LBCU) = (L'B'C'U)$; hence the ranges $(LBCU)$ and $(L'B'C'U)$ are in perspective. Hence LL', BB', CC' meet in a point, i.e. the triangles ABC, $A'B'C'$ are homological.

Let the triangles ABC, $A'B'C'$ be homological, then they are reciprocal for a conic. For let BC and $A'C'$ meet in M. [Suppose a conic exist for which ABC, $A'B'C'$ are reciprocal. To construct this conic by XXV. 12, we must find a self-

conjugate triangle and a pole and polar. Now the polar of B is $A'M$ and the polar of A' is BM. Hence $A'BM$ is self-conjugate. Also A is the pole of $B'C'$.] By XXV. 12 describe a conic such that the triangle $A'BM$ is self-conjugate for it, and that A is the pole of $B'C'$.

Then A' is the pole of BC, B is the pole of $A'C'$, and A is the pole of $B'C'$. Hence C' is the pole of AB. Now let the polar of C cut $C'B'$ in B''. Then the triangles ABC and $A'B''C'$ are reciprocal and therefore homological. Hence AA', BB'', CC' meet in a point. But AA', BB', CC' meet in a point. Hence B' and B'' coincide, i.e. the triangles ABC, $A'B'C'$ are reciprocal for the above conic.

Notice that *if two triangles are reciprocal for a conic, the centre of homology of the triangles is the pole of the axis of homology;* for the pole of AA' which passes through the centre of homology is U which lies on the axis of homology, so for BB'.

Given a triangle ABC and a conic a, we can describe the reciprocal triangle $A'B'C'$, and then determine the centre O and axis s of perspective of the triangles ABC, $A'B'C'$. It is convenient to call *O the pole and s the polar of the triangle ABC for the conic a.*

Ex. 1. *BC, CA, AB meet any conic in XX', YY', ZZ', and the conic meets AX again in L, AX' in L', BY in M, BY' in M', CZ in N, CZ' in N'. Show that LL', MM', NN', meet BC, CA, AB on a line.*

For, by the quadrangle construction, the polar $B'C'$ of A passes through $(LL'; BC)$.

Ex. 2. *Any triangle inscribed in a conic and the triangle formed by the tangents at the vertices are homological.*

Ex. 3. Hesse's theorem. *If the opposite vertices AA' and the opposite vertices BB' of a complete quadrilateral be conjugate for the same conic, then the opposite vertices CC' are also conjugate for this conic.* (See also XXIX. 4, Ex. 9.)

Let the triangle reciprocal to the triangle ABC for the conic be PQR. Then QR passes through A', since A and A' are conjugate. So RP passes through B'. Hence PQ passes through C'; for the triangles ABC and PQR are homological. Hence C and C' are conjugate.

Ex. 4. *If two pairs of opposite sides of a complete quadrangle be conjugate for the same conic, then the third pair is also conjugate for this conic.*

Reciprocate Hesse's theorem.

Ex. 5. *The points PP', QQ', RR' divide harmonically the diagonals AA', BB', CC' of a quadrilateral; show that the six points P, P', Q, Q', R, R' lie on a conic.*

Draw a conic through P, P', Q, Q', R. Then for this conic A, A' are conjugate since (AA', PP') is harmonic; so B, B' are conjugate. Hence by Hesse's theorem C, C' are conjugate. Hence the conic passes through R' since (CC', RR') is harmonic.

XIV

1. Through a given point O on a conic is drawn any line cutting the conic at P and the sides of a given inscribed triangle at X, Y, Z. Show that the cross ratio (XY, ZP) is constant.

2. An infinite number of triangles can be described, having the same circumscribing, nine-points and polar circles as a given triangle.

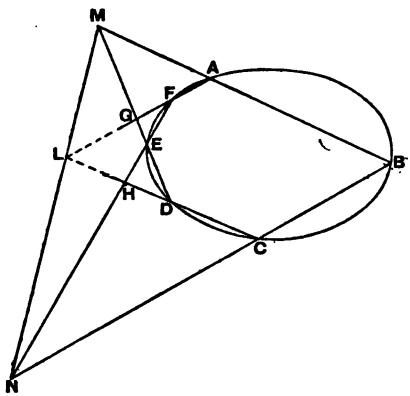
CHAPTER XV

PASCAL'S THEOREM AND BRIANCHON'S THEOREM

Pascal's Theorem.

1. *The meets of opposite sides of a hexagon (six-point) inscribed in a conic are collinear.*

Let the six points be A, B, C, D, E, F. Let the opposite sides AB, DE meet in M, and the opposite sides BC, EF



meet in N. Let AF meet MD in G, and let CD meet NF in H. Then we have to show that MN, FG, HD are concurrent. This is true if $(EMGD) = (ENFH)$, for the ranges, having a common point, will be in perspective; i.e. if

$$A(EBFD) = C(EBFD),$$

which is true. Hence the meet M of AB, DE, the meet N of BC, EF, and the meet L of CD, FA are collinear.

Conversely, *if the meets of opposite sides of a hexagon (six-point) are collinear, the six vertices lie on a conic.*

For if LMN are collinear, we have $(EMGD) = (ENFH)$.

Hence $A(EBFD) = C(EBFD)$. Hence A, B, C, D, E, F lie on the same conic.

The line LMN is called the *Pascal line* of the six-point $ABCDEF$. Observe that for every different order of the points A, B, C, D, E, F we get a different Pascal line.

Notice that if two consecutive points, e.g. B and C, coincide, the side BC becomes the tangent at B or C.

For another proof see XXIX. 4.

Ex. 1. *In every hexagon inscribed in a conic, the two triangles formed by taking alternate sides are copolar.*

Ex. 2. *Six points on a conic determine* 60 *hexagons inscribed in the conic.*

For having chosen our starting-point, the other vertices can be chosen in $5.4.3.2 \div 2$ ways, for $ABCDEF$ is the same as $AFEDCB$.

Ex. 3. *The* 60 *Pascal lines belonging to six given points on a conic intersect three by three; and also four by four.*

Let the homological triangles of any one hexagon be XYZ, $X'Y'Z'$. Then XX', YY', ZZ' meet in a point. Also XX' is the Pascal line of $CDEBAF$, YY' of $ABCFED$, ZZ' of $BCDAFE$.

Again the Pascal lines of $ABCDEF$, $ABFDEC$, $ABCEDF$ and $ABFEDC$ meet at $(AB; DE)$.

Ex. 4. *A, B, C, D, E are any five points. EA, BC meet in A'; AB, CD meet in B'; BC, DE meet in C'; CD, EA meet in D'; DE, AB meet in E'; and AD, BC meet in F. Show that FB' touches the conic through $A'B'C'D'E'$.*

We want to form a Pascal hexagon. Start from A'; then we must go to C' or D', to C', say. Then we must go to E', B', B', D', A' where $B'B'$ is to be FB'. We have to show that $(A'C'; B'B')$, $(C'E'; B'D')$, $(E'B'; D'A')$ are collinear, i.e. that F, D, A are collinear.

Ex. 5. *ABC, $A'B'C'$ are coaxal triangles; AC and $A'B'$ meet in P, AB and $A'C'$ meet in Q; show that $BCB'C'PQ$ are on a conic.*

Ex. 6. *The chord QQ' of a conic is parallel to the tangent at P, and the chord PP' is parallel to the tangent at Q; show that PQ and $P'Q'$ are parallel.*

Consider $PPP'Q'QQ$.

Ex. 7. *The tangents at the vertices of a triangle inscribed in a conic meet the opposite sides in three collinear points.*

Ex. 8. *PQ, PR are chords of a parabola. PR meets the diameter through Q in V, and PQ meets the diameter through R in U; show that UV is parallel to the tangent at P.*

Consider $PPR\Omega\Omega Q$, where Ω is the point at infinity on the parabola.

2. Since Pascal's theorem is true for a hyperbola however near the hyperbola approaches two lines, it is true for two lines, the six points being situated in any manner on the two lines.

But each case may be proved as in § 1.

Ex. *If any four-sided figure be divided into two others by a line, the three meets of the internal diagonals are collinear.*

Let the four-sided figure $ABCD$ be divided into two others $ABFE$, $EFCD$. Now apply Pascal's theorem to $ACEBDF$.

3. *If OQ and OR be the tangents of a conic at Q and R, and if P be any point on the conic, then PQ and PR cut any line through O in points which are conjugate for the conic.*

Let PQ and PR cut any line through O in F and G. Let FR and GQ meet in U. Consider the six-point $PQQURR$, the points Q, Q being on the tangent at Q and the points R, R on the tangent at R. Then since the meets F, O, G of opposite sides are collinear, the six points lie on a conic. But five points lie on the given conic; hence the sixth point U also lies on the given conic. Hence F and G are two harmonic points of the inscribed quadrangle $PQUR$. Hence F and G are conjugate points.

Conversely, *if any two conjugate points lying on a line through O be joined to the points of contact of the tangents from O, then the joining lines meet on the conic.*

Let F and G be conjugate points on a line through O. Join FQ cutting the conic again in P, and join PR cutting FG in G'. Then F and G' are conjugate, and also F and G. Hence G' coincides with G; i.e. FQ and GR meet on the conic. So FR and GQ meet on the conic.

Ex. 1. *If PP' be conjugate points for a central conic, and QQ' be the ends of the diameter which bisects chords parallel to PP'; show that PQ, $P'Q'$ cut on the conic, and so do PQ', $P'Q$.*

Here O is at infinity.

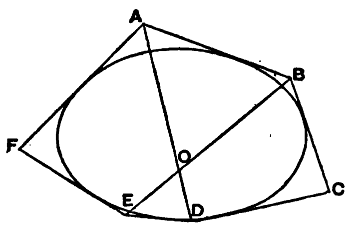
Ex. 2. *If P and P' be conjugate points lying on a diameter of a hyperbola, show that parallels to the asymptotes through P and P' cut again on the curve.*

Here Q and R are at infinity.

Ex. 3. *The diameter bisecting the chord QQ' of a parabola cuts the curve in P, and RR' are points on this diameter equidistant from P; show that the other lines joining $QQ'RR'$ meet on the curve.*

Brianchon's Theorem.

4. *The joins of opposite vertices of a hexagon (six-side) circumscribing a conic are concurrent.*



To prove Brianchon's theorem, we reciprocate Pascal's theorem.

In the figure AB, BC, CD, DE, EF, FA are the six sides touching the conic. The theorem is that AD, BE, CF are concurrent.

The point O is called the *Brianchon point* of the hexagon $ABCDEFA$.

Notice that when two of the sides, e. g. CD and DE, coincide, the point D becomes the point of contact of either CD or DE.

Ex. 1. *In every hexagon circumscribed to a conic, the two triangles formed by taking alternate vertices are coaxal.*

Ex. 2. *Six tangents to a conic determine* 60 *hexagons circumscribed to the conic.*

Ex. 3. *The* 60 *Brianchon points belonging to six given tangents to a conic are collinear three by three; and also four by four.*

Reciprocate.

Ex. 4. *The hexagon formed by the six lines in order obtained by joining alternate pairs of vertices of a Brianchon hexagon is a Pascal hexagon.*

Ex. 5. *Reciprocate Ex. 4.*

Ex. 6. *$ABCDA$ is a quadrilateral circumscribing a parabola; show that the parallel through A to CD and the parallel through C to DA meet on the diameter through B.*

Let the straight line at infinity touch the parabola at Ω and cut AD at F and CD at G; and consider the hexagon $ABCG\Omega FA$.

Ex. 7. *$ABCDEA$ is a pentagon circumscribing a parabola; show that the parallel through A to CD, and the parallel through B to DE meet on CE.*

Ex. 8. *The lines BC, CA, AB touch a conic at L, M, N; show that AL, BM, CN are concurrent.*

Ex. 9. *The line $C'B'A$ touches a conic in P, ACB touches in P', $B'CA'$ touches in Q and $C'BA'$ in Q'. Show that $A'P'$, AQ meet on CC'.*

Consider $AP'CQA'C'A$.

5. *If OQ and OR be the tangents of a conic at Q and R, and if any tangent meet OQ, OR in K, L, then the joins of K and L to any point E on QR are conjugate lines; and, conversely, if through any point E on QR any two conjugate lines be drawn cutting OQ in M, K and OR in L, N, then MN and KL touch the conic.*

To prove this proposition, reciprocate the proposition of § 3.

Ex. 1. *Parallel to a diameter of a conic are drawn a pair of conjugate lines; show that the diagonals of the parallelogram formed by these lines and the tangents at the ends of the diameter touch the conic.*

Ex. 2. *Two parallel lines which are conjugate for a hyperbola meet the asymptotes in points such that the other lines joining them touch the curve.*

Ex. 3. *Through a point on the chord of contact PQ of the tangents from T to a parabola are drawn parallels to TP and TQ meeting TQ and TP in R and U; show that RU touches the parabola.*

XV

1. Two triangles are inscribed in a conic. The sides of the one meet the sides of the other in nine points. Show that any connector of two of these nine points is a Pascal line of the six vertices of the triangles, unless it is one of the sides of the triangles.

2. The triangles ABC, $A'B'C'$ are in plane perspective. BC meets $A'B'$ at Y and $A'C'$ at Z', CA meets $B'C'$ at Z and $B'A'$ at X', and AB meets $C'A'$ at X and $C'B'$ at Y'. Show that
$$BY.BZ'.CZ.CX'.AX.AY' = CY.CZ'.AZ.AX'.BX.BY'.$$

3. AA', BB', CC' are the diagonals of a complete quadrilateral, A', B', C' being collinear points. AO meets BC at M, CO meets AB at L, LM meets $B'C'$ at N and AC at P. If PB and ON meet at R, show that R is the remaining intersection of the conics $OBB'AA'$ and $OBB'CC'$ and that OR is the tangent at O to the conic $OCC'AA'$.

4. Three angles have collinear vertices. Show that their six legs intersect in twelve other points, which can be divided in four ways into a Pascal hexagon and a Brianchon hexagon.

5. If two triangles be the reciprocals of one another for a conic α, the intersections of non-corresponding sides lie on a conic β and the connectors of non-corresponding vertices touch a conic γ; and β and γ are reciprocal for α. If one triangle be inscribed in the other, the three conics coincide.

6. If a hexagon can be inscribed in one conic and circumscribed to another, the Brianchon point is the pole of the Pascal line for each conic.

7. If the tangents of a parabola at P and Q cut at T, and on the diameter through P there be taken any point R, show that RT is conjugate to the parallel through R to the tangent at Q.

8. The lines BC, CA, AB touch a conic at A', B', C'. Through B is drawn any line meeting $A'B'$ at Z and $B'C'$ at X, and AZ meets $C'A'$ at Y. Show that the triangle XYZ is inscribed in the triangle $A'B'C'$ and circumscribed to the triangle ABC and is also self-conjugate for the conic.

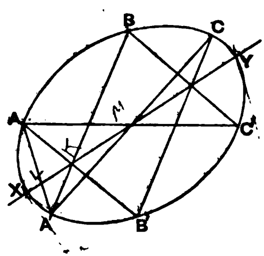
CHAPTER XVI

HOMOGRAPHIC RANGES ON A CONIC

1. Two systems of points $ABC...$ and $A'B'C'...$ on a conic are said to be *homographic ranges on the conic* when the pencils $P(ABC...)$ and $Q(A'B'C'...)$ are homographic, P and Q being points on the conic. Hence two ranges on a conic which are homographic subtend, at any points on the conic, pencils which are homographic.

To construct homographic ranges on a conic, take two homographic pencils at points P and Q on the conic; the rays of these pencils will determine on the conic two homographic ranges. Given one of these pencils, three rays of the other pencil may be taken arbitrarily. Hence given a range of points on a conic, in constructing a homographic range on the conic, three points may be taken arbitrarily.

2. *If $(ABC...)$ and $(A'B'C'...)$ be two homographic ranges on a conic, then the meet of AB' and $A'B$, of BC' and $B'C$, and generally of PQ' and $P'Q$, where PP', QQ' are any two pairs of corresponding points, all lie on a line* (called the *homographic axis*).



First consider all the meets which belong to A and A'. These all lie on a line. For

$$A(A'B'C'...) = A'(ABC...).$$

Hence all the meets $(AB'; A'B)$, $(AC'; A'C)$, $(AD'; A'D)$, ... lie on an axis. So all the meets which belong to B and B' lie on an axis. So for CC', DD', ...

We have now to prove that all these axes are the same. The inscribed six-point $AB'CA'BC'$ shows that the meets

$(AB'; A'B)$, $(B'C; BC')$, $(CA'; C'A)$ are collinear. Now $(AB'; A'B)$ and $(CA'; C'A)$ determine the axis of AA'; so $(AB'; A'B)$ and $(B'C; BC')$ determine the axis of BB'. Hence the axes of AA' and BB' coincide; i.e. every two axes, and therefore all the axes, coincide. Hence all the *cross meets* $(PQ'; P'Q)$ lie on the same line.

Conversely, *if all the cross meets of two ranges on a conic are collinear, these ranges are homographic.*

For, if $(AB'; A'B)$, $(AC'; A'C)$, ... are collinear, then $A'(ABC\ldots) = A(A'B'C'\ldots)$, i.e. $(ABC\ldots) = (A'B'C'\ldots)$.

3. *Given three pairs of corresponding points ABC, $A'B'C'$ of two homographic ranges on a conic, to construct the point D' corresponding to D.*

The meets $(AB'; A'B)$ and $(AC'; A'C)$ give the homographic axis; and we know that $(AD'; A'D)$ is on the homographic axis. Hence the construction—Let $A'D$ cut the homographic axis in δ, join $A\delta$, cutting the conic again in the required point D'.

4. *Two homographic ranges on a conic have two common points, viz. the points where the homographic axis cuts the conic.*

Let the homographic axis cut the conic in X and Y. To get the point X' corresponding to X, we join A' to X cutting XY in X and then join AX cutting the conic again in X'. Hence X' is X. So Y' is Y.

And there can be no common point other than X and Y. For if D and D' coincide, then each coincides with δ. Hence D, D' and δ must be at X or Y.

If E, F are the common points of the homographic ranges $(ABC\ldots)$ and $(A'B'C'\ldots)$ on a conic, then

$$(EF, AA') = (EF, BB') = \ldots;$$

and, conversely, if E and F are fixed points on a conic and P and P' variable points on the conic such that (EF, PP') is constant, then P and P' generate homographic ranges on the conic.

Consider the pencils subtended at any point on the conic.

5. Reciprocally, *two homographic sets of tangents to a conic can be formed by dividing two tangents homographically in ABC ... and $A'B'C'$ ...; then the second tangents from ABC ... will form a set of tangents homographic with the second tangents from $A'B'C'$ ....*

For any tangent will cut the two sets in homographic ranges.

Again, *all the cross joins will pass through a point called the homographic pole; and the tangents from the homographic pole will be the self-corresponding lines in the two sets of homographic tangents.*

This follows by Reciprocation from the previous articles.

Ex. 1. *The points of contact of two homographic sets of tangents are homographic ranges; and, conversely, the tangents at points of two homographic ranges on a conic form homographic sets of tangents.*

For tangents are homographic with their points of contact.

Ex. 2. *The pencils $A(PQR\ ...)$ and $A'(PQR\ ...)$ are homographic. A line meets AP in p, $A'P$ in p', and so on. Show that there are two positions of the line such that*

$$pp' = qq' = rr' = \ldots.$$

Viz. the asymptotes of the conic through $AA'PQR$ ....

Ex. 3. *The joins of corresponding points of two homographic ranges on a conic touch a conic having double contact with the given conic at the common points of the given ranges.*

Let AA' cut XY in L, the tangent at X in a, and the tangent at Y in a'; let BB' cut XY in M, the tangent at X in b, and the tangent at Y in b'. Let AB', $A'B$ cut XY in K. Then

$$(ALA'a) = X(ALA'a) = (AYA'X) = (BYB'X)$$

[since X, Y are the common points]

$$= Y(BYB'X) = (Bb'B'M) = (B'MBb').$$

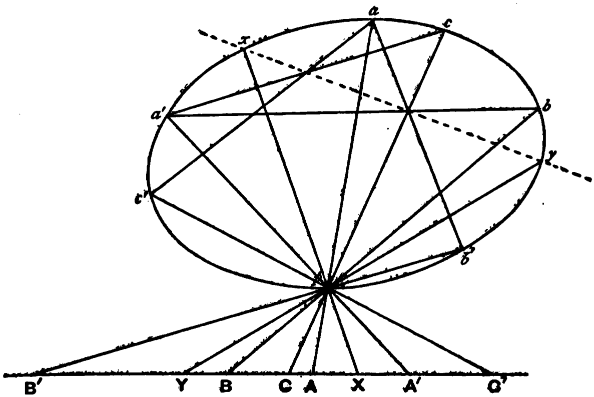
Now AB', LM and $A'B$ meet in K. Hence ab' passes through K. So $a'b$ passes through K. Hence XY, ab', $a'b$ are concurrent. Hence, by Brianchon, a conic touching the conic at X and at Y and touching AA' will also touch BB', and similarly CC', &c. (See also XXIX. 10.)

6. *Given a conic and a ruler, construct the common points of two homographic ranges on the same line.*

Let the ranges be ABC ... and $A'B'C'$ .... Take any point p on the conic, and let pA, pA', pB, pB', ... cut the conic again in a, a', b, b', ... The ranges abc ... and $a'b'c'$ ... on the conic are homographic; for

$$(abc \ldots) = p\,(abc \ldots) = (ABC \ldots) = (A'B'C' \ldots)$$
$$= p\,(A'B'C' \ldots) = (a'b'c' \ldots).$$

Now determine the homographic axis of the ranges $(abc \ldots)$ and $(a'b'c' \ldots)$ by connecting the cross meets $(ab'; a'b)$, &c.; and let this axis cut the conic in x and y. Then if px and py cut AB in X and Y, X and Y are the common points of the ranges ABC ... and $A'B'C'$ ....



For

$$(XYABC \ldots) = p\,(XYABC \ldots) = (xyabc \ldots) = (xya'b'c' \ldots)$$
$$= p\,(xya'b'c' \ldots) = (XYA'B'C' \ldots);$$

i.e. XY correspond to themselves in the ranges ABC ... and $A'B'C'$ ....

Given a conic and a ruler, construct the common rays of two homographic pencils having the same vertex.

Join the vertex to the common points of the ranges determined by the pencils on any line.

Ex. *Given two pairs of corresponding points of two homographic ranges and one common point, construct the other common point.*

CHAPTER XVII

RANGES IN INVOLUTION

1. If we take pairs of corresponding points, viz. AA', BB', CC', DD', EE', ... on a line, such that a cross ratio of any four of these points (say AD', $C'E$) is equal to the corresponding cross ratio of the corresponding points (viz. $A'D$, CE'), then the pairs of points AA', BB', CC', ... are said to be in involution or to form an involution range.

Or more briefly—If the ranges $(AA'BB'CC'...)$ and $(A'AB'BC'C...)$ are homographic, then the pairs of points AA', BB', CC',... are in involution.

To avoid the use of the vague word 'conjugate' let us call each of a pair of corresponding points, AA' say, the *mate* of the other, so that A is the mate of A' and A' is the mate of A. Let us call AA' together a pair of the involution.

There is no good notation for involution. The notation we have used above implies that A and B are related to one another in a way in which A and B' are not related; and this is not true. If we use the notation AB, CD, EF, ... for pairs of points in involution, this objection disappears; but there is now nothing to tell us that A and B are corresponding points.

2. The following is the fundamental proposition in the subject and enables us to recognize a range in involution.

If two homographic ranges, viz.

$$(AA'BCD\,...)\ \textit{and}\ (A'AB'C'D'...),$$

be such that to one point A corresponds the same point, viz. A', whichever range A is supposed to belong to, the same is true of every other point, and the pairs of corresponding points AA', BB', CC', DD', ... are in involution.

We have to prove that

$$(AA'BB'CC'DD'...) = (A'AB'BC'CD'D\,...),$$

given that $\quad (AA'BCD\,...) = (A'AB'C'D'...).$

Now if P be considered to belong to the first range, its mate P' in the second range is determined by the equation

$$(AA'BP) = (A'AB'P').$$

Let P be B', then the mate P' of B' is given by the equation $(AA'BB') = (A'AB'P')$. Now we have identically $(AA'BB') = (A'AB'B)$. Hence P' is B. Hence B has the same mate, viz. B', whichever range it is considered to belong to. Again, we may consider the homography to be determined by the equation $(AA'CP) = (A'AC'P')$; hence, as before, C has the same mate in both ranges. Similarly every point has the same mate in both ranges, i.e.

$$(AA'BB'CC'\ldots) = (A'AB'BC'C\ldots).$$

The commonest case of this proposition is—

If $$(AA'BC) = (A'AB'C');$$
then AA', BB', CC' are in involution.

Two pairs of points determine an involution.

For the pairs of points PP' which satisfy the relation $(AA'BP) = (A'AB'P')$ are in involution.

Notice that *an involution range projects into an involution range;* for the homographic ranges $(AA'BB'\ldots)$ and $(A'AB'B\ldots)$ project into homographic ranges.

Ex. 1. *If (CB, AA') and $(C'B', AA')$ be harmonic, then (AA', BB', CC') are in involution.*

For $(CB, AA') = (C'B', A'A)$.

Ex. 2. *If* $(CA, A'B') = (AB, A'C') = -1$, *then* (AA', BB', CC') *is an involution.*

Ex. 3. *If* $(AA', BC) = (BB', CA) = (CC', AB) = -1$, *show that* $(A'A, B'C') = (B'B, C'A') = (C'C, A'B') = -1$, *and that* $(AA', BC', B'C)$, $(BB', AC', A'C)$ *and* $(CC', AB', A'B)$ *are involutions.*

Project the range so that A goes to infinity.

3. *If AA', BB', CC' be three pairs of points in involution, the following relations are true, viz.*

$$AB'.BC'.CA' = -A'B.B'C.C'A,$$
$$AB'.BC.C'A' = -A'B.B'C'.CA,$$
$$AB.B'C'.CA' = -A'B'.BC.C'A,$$
$$AB.B'C.C'A' = -A'B'.BC'.CA.$$

Take any one of the relations, viz.

$$AB . B'C' . CA' = -A'B' . BC . C'A.$$

This is true if $AB/BC \div AC'/1 = -A'B'/B'C' \div A'C/1$,

i.e. if $\quad AB/BC \div AC'/C'C = A'B'/B'C' \div A'C/CC'$,

i.e. if $\quad (AC, BC') = (A'C', B'C)$.

And this is true; hence the relation in question is true.

Similarly the other relations can be proved.

Conversely, *if any one of these relations be true, then AA', BB', CC' are in involution.*

For suppose $AB . B'C' . CA' = -A'B' . BC . C'A$; then as above $(AC, BC') = (A'C', B'C)$; hence AA', BB', CC' are in involution.

Ex. *If (AA', BB', CC') be in involution, then*

$$(A'A, BC) . (B'B, CA) . (C'C, AB) = -1.$$

4. *If AA', BB', CC', ... be in involution, and if any fixed pair of corresponding points UU' be taken as origins, and if PP' be any variable pair of corresponding points, then*

$$UP . UP' \div U'P . U'P'$$

is constant.

It will be sufficient to prove that

$$UP . UP' \div U'P . U'P' = UA . UA' \div U'A . U'A',$$

where AA' is a fixed pair of corresponding points. This is true if $PU/UA \div PU'/U'A = P'U'/U'A' \div P'U/UA'$, i.e. if $(PA, UU') = (P'A', U'U)$. And this is true; hence the relation in question is true.

Particular cases of this theorem are—

$$AB . AB' \div A'B . A'B' = AC . AC' \div A'C . A'C',$$
$$CA . CA' \div C'A . C'A' = CD . CD' \div C'D . C'D'.$$

Conversely, *if UU' be fixed points, and if PP' be variable points such that $UP . UP' \div U'P . U'P'$ is constant; then PP' generate an involution in which UU' are corresponding points.*

For take any point A and let A' be the position of P' when P is at A. Then

$$UP . UP' \div U'P . U'P' = UA . UA' \div U'A . U'A';$$

hence $(PA, UU') = (P'A', U'U)$, i.e. P and P' are corre-

sponding points in the involution determined by the two pairs AA', UU'.

5. *In an involution range, if any two of the segments AA', BB', ... bounded by corresponding points overlap, then every two overlap; and if any two do not overlap, then no two overlap.*

For suppose AA' and BB' overlap, then any two others CC' and DD' overlap.

$$\underline{A \qquad B \quad A' \qquad B'}$$

For $$\frac{AB.AB'}{A'B.A'B'} = \frac{AC.AC'}{A'C.A'C'}.$$

But since AA' and BB' overlap, the sign of

$$AB.AB' \div A'B.A'B'$$

is $-$. Hence the sign of $AC.AC' \div A'C.A'C'$ is $-$. Hence AA' and CC' overlap; for if AA' and CC' do not overlap, the sign of this expression is $+$, as we see from the figures—

$$\underline{A \qquad A' \qquad C \qquad C'} \qquad \underline{A \qquad C \qquad C' \qquad A'}$$

We have shown that if AA' and BB' overlap, then AA' and CC' overlap. Hence, since CC' and AA' overlap, it follows that CC' and DD' overlap, i.e. that every two such segments overlap.

Conversely, suppose AA' and BB' do not overlap, then CC' and DD' do not overlap; for if they do overlap, by the first part of the proof it follows that AA' and BB' overlap.

6. The *centre of an involution range* is the point corresponding to the point at infinity.

If O be the centre of the involution of which P and P' are a pair of corresponding points, then $OP.OP'$ is constant; and, conversely, if a pair of points P and P' be taken on a line, such that $OP.OP'$ is constant, then P and P' generate an involution range of which O is the centre.

Let O be the centre of the involution range (AA', BB', PP', ...). Then Ω' being the point at infinity upon the line, we have by definition

$$(O\Omega' AA'BB'PP') = (\Omega' OA'AB'BP'P);$$

$\therefore$ $(O\Omega', AP) = (\Omega' O, A'P')$;

$\therefore$ $OA/A\Omega' \div OP/P\Omega' = \Omega' A'/A'O \div \Omega' P'/P'O$,

and $A\Omega' = P\Omega'$ and $\Omega' A' = \Omega' P'$;

$\therefore$ $OP.OP' = OA.OA'$, which is constant.

Conversely, if $OP.OP'$ be constant, let A' be the position of P' when P is at A. Then we have $OP.OP' = OA.OA'$. Hence by writing the above steps backward we get

$$(O\Omega' AP) = (\Omega' OA'P'),$$

where Ω' is the point at infinity on the line. Hence P and P' are a pair of corresponding points in the involution determined by $(O\Omega', AA')$, i.e. P and P' generate an involution of which O is the centre.

Ex. 1. *If O be the centre of the involution $(AA, BB', CC', \ldots)$, show that* $AB.AB' \div A'B.A'B' = AO \div A'O$.

To prove this, make the relation projective by introducing infinite segments in such a manner that the same letters occur on each side of the relation. We get

$$AB.AB' \div A'B.A'B' = AO.A\Omega' \div A'O.A'\Omega',$$

and this is a particular case of the theorem

$$AB.AB' \div A'B.A'B' = AC.AC' \div A'C.A'C'.$$

Ex. 2. *Show that* $OA : OB :: AB' : BA'$.

Ex. 3. *If α bisect AA' and β bisect BB', show that*

(*a*) $2.AO.\alpha\beta = AB.AB'$;

(*b*) $4.\alpha O.\alpha\beta = AB.AB' + A'B.A'B'$.

To verify any relation connecting points in involution, we take O as origin and use the fact that $aa' = bb' = \ldots = k$, putting $a' = k/a$, $b' = k/b$, ...

Ex. 4. *Prove that* $AA' = \dfrac{AB.A'C}{BC} + \dfrac{AB'.A'C'}{B'C'}$.

Project A to infinity; then A' becomes O.

Ex. 5. *If P be any point on the line of the involution AA', BB', CC', prove that*

$$\frac{CA}{C'A'} . BC'.PA' + \frac{CB}{C'B'} . C'A.PB' = AB.PC.$$

Ex. 6. *Prove that* $\dfrac{CA}{C'A'} \cdot BC' + \dfrac{CB}{C'B'} \cdot C'A = AB$.

This is not a projective relation; so we use the method of Ex. 3.

Ex. 7. *Any two homographic ranges can be placed in two ways so as to be in involution.*

Viz. by placing I on J' and placing A and A' on the same or opposite sides of O which is now I and J'.

Ex. 8. *Of these two involutions one is overlapping and the other not.*

Consider $O\Omega'$, AA'.

Ex. 9. *If any four points on a line be inverted for any point on this line, every cross ratio remains unchanged.*

For if $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = OD \cdot OD'$, then (AA', BB', CC', DD') are in involution. Hence

$$(AA'BB'CC'DD') = (A'AB'BC'CD'D).$$

Hence $(A'B'C'D') = (ABCD)$.

7. A point on the line of an involution range which coincides with its mate is called a *double point* (or focus) of the involution.

An involution range has two, and only two, double points; and the segment joining the double points is bisected by the centre and divides the segment joining any pair of corresponding points harmonically.

If AA', PP' be two pairs of corresponding points of an involution whose centre is O, we have seen that

$$OP \cdot OP' = OA \cdot OA'.$$

Suppose P and P' coincide in E. Then $OE^2 = OA \cdot OA'$, hence $OE = \pm \sqrt{OA \cdot OA'}$. Hence there are two double points, E and F say, which are equidistant from O. Also, since $OE^2 = OF^2 = OA \cdot OA'$ and O bisects EF, it follows that (AA', EF) is harmonic, i.e. EF divides the segment joining any two corresponding points harmonically.

Conversely, *if the pairs of points AA', BB', ... on the same line are such that points E, F can be found which are harmonic with every pair, then (AA', BB', CC') is an involution, of which E and F are the double points.*

For bisect EF at O. Then since (EF, AA') is harmonic, $OA \cdot OA' = OE^2$. So $OB \cdot OB' = OE^2$ and so on. Hence $OE^2 = OF^2 = OA \cdot OA' = OB \cdot OB' = \ldots$.

Notice that *the centre is always real*, being the mate of the point at infinity. But the double points will be imaginary when $OA \,.\, OA'$ is negative, i. e. when O lies between A and A'. *If the double points coincide*, the centre must coincide with them for O bisects EF. Also since $OA \,.\, OA' = OE^2 = 0$ in this case, one of each pair of corresponding points A, A' must be at O and the other may be anywhere. Conversely, *if we take any number of points A, B, C, ... all coincident, and any points A', B', C', ... such that A, A', B', C', ... are collinear, then $(AA', BB', CC', ...)$ is an involution;* for A, A' are harmonic with E, F if E, F are both at A, since three coincident points are harmonic with any other point.

8. *The double points of an overlapping involution are imaginary and those of a non-overlapping involution are real.*

Take O the centre of the involution. Then

$$OA \,.\, OA' = OB \,.\, OB' = \dots = OE^2 = OF^2.$$

Now in an overlapping involution $O\Omega'$ and AA' overlap, i. e. O lies between A and A'. Hence $OA \,.\, OA'$ is negative, i. e. OE^2 and OF^2 are negative, i. e. E and F are imaginary.

Similarly in a non-overlapping involution, OE^2 and OF^2 are positive, i. e. E and F are real.

An overlapping involution is sometimes called a *negative involution* and a non-overlapping involution is called a *positive involution.*

Ex. 1. *If E and F be the double points of $(AA', BB', CC', ...)$, show that $AB \,.\, AB' \div A'B \,.\, A'B' = AE^2 \div A'E^2$.*

Ex. 2. *Also $AB' \,.\, BE \,.\, EA' = -A'B \,.\, B'E \,.\, EA$.*

Ex. 3. *Also $EF^2 \,.\, \alpha\beta^2 = AB \,.\, AB' \,.\, A'B \,.\, A'B'$.*

Take O as origin. Then

$$e^2 = aa' = bb' = \dots \text{ and } EF = -2e.$$

Ex. 4. *If the segments AA', BB', ... joining corresponding points have the same middle point, show that AA', BB', ... form an involution; and find the centre and double points.*

Ω' the point at infinity and E the middle point are harmonic with every segment AA'. Hence Ω', E are the double points and Ω' is the centre.

Ex. 5. *If AA', BB' be pairs of points in an involution, one of whose double points is at infinity, then $AB = -A'B'$.*

Ex. 6. *If any transversal through V (the internal vertex of the harmonic triangle of a quadrilateral circumscribing a conic) cut the sides in AA', BB' and the conic in PP'; show that (AA', BB', PP') is an involution, the double points being V and the meet of UW with the transversal.*

Ex. 7. *Through a given point O draw a line meeting two conics (or two pairs of lines) in points AA', BB' such that $(OAA'BB') = (OA'AB'B)$.*

Join O to the meet of the polars of O.

9. *A system of coaxal circles is cut by any transversal in pairs of points in involution.*

For if the transversal cut the circles in AA', BB', CC', ... and the radical axis in O, then

$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = \dots .$$

Hence AA', BB', CC', ... form an involution of which O is the centre.

Ex. 1. *Give a geometrical construction for the double points of the involution determined on a line by a system of coaxal circles.*

Ex. 2. *A line touches two circles in A and A' and cuts a coaxal circle in B and B'; show that (AA', BB') is harmonic.*

Ex. 3. *Of the involution determined by a system of coaxal circles on the line of centres, the limiting points are the double points.*

Ex. 4. *Any line through the radical centre of three circles cuts them in pairs of points in involution.*

Ex. 5. *If a line meet three circles in three pairs of points in involution, then either the circles are coaxal or the line passes through their radical centre.*

10. *If EF be the double points of an involution of which AA' and BB' are any two pairs of corresponding points, then $(AB', A'B, EF)$ are in involution, and so are $(AB, A'B', EF)$.*

For $(AB', A'B, EF)$ are in involution if

$$(ABEF) = (B'A'FE), \text{ i.e. } = (A'B'EF);$$

and this is true, for E corresponds to itself and so does F.

Similarly $(AB, A'B', EF)$ are in involution.

Notice that we have incidentally proved that *if E, F are the common points of the two homographic ranges $(ABC\ldots)$ and $(A'B'C'\ldots)$, then $(AB', A'B, EF)$ is an involution.*

Ex. *If in two homographic ranges, the common points are harmonic with any one pair of corresponding points, the ranges are in involution.*

For if $(EF, AA') = -1$, $(EF, AA') = (EF, A'A)$. Hence $(EFAA'BC\ldots) = (EFA'AB'C'\ldots)$. Hence A, A' are interchangeable.

***11.** *If AA', BB', CC' be pairs of points in involution, and if P, Q, R be the middle points of AA', BB', CC', show that*

$$A'A^2 . QR + B'B^2 . RP + C'C^2 . PQ + 4\,PQ . QR . RP = 0;$$

and if U be any point on the same line, then

$$(AU^2 + A'U^2)\,QR + (BU^2 + B'U^2)\,RP + (CU^2 + C'U^2)\,PQ$$
$$= -4\,PQ . QR . RP.$$

Take the centre of the involution as origin and use abridged notation; then if $OA' = a_1$, and so on,

$$A'A^2 = (a - a_1)^2 = a^2 + a_1{}^2 - 2aa_1 = (a + a_1)^2 - 4aa_1.$$

But $\quad a + a_1 = 2p$ and $QR = r - q$,

and $\quad aa_1 = bb_1 = cc_1 = \lambda$, say;

$$\therefore\ A'A^2 . QR = (4p^2 - 4\lambda)(r - q);$$
$$\therefore\ \Sigma\,(A'A^2 . QR) = 4\,\Sigma p^2 (r - q) - 4\lambda\,\Sigma\,(r - q)$$
$$= -4\,(q - p)(r - q)(p - r)$$
$$= -4\,PQ . QR . RP.$$

Again, if x be the distance of U from the origin

$$AU^2 = (x - a)^2.$$

Hence $\quad \Sigma\,\{(AU^2 + A'U^2)\,QR\}$

$$= \Sigma\,\{[2x^2 - 2x(a + a_1) + a^2 + a_1{}^2](r - q)\}$$
$$= 2x^2\,\Sigma\,(r - q) - 4x\,\Sigma p\,(r - q)$$
$$+ \Sigma\,\{a^2 + a_1{}^2 - 2aa_1 + 2\lambda\}\,(r - q)$$
$$= \Sigma A'A^2 . QR$$
$$= -4PQ . QR . RP \text{ by the former part.}$$

Ex. 1. *With the same notation, show that*

$$AB . AB'/AC . AC' = PQ/PR.$$

Ex. 2. *Also, X being any point on the same line,*

$$XA . XA' . QR + XB . XB' . RP + XC . XC' . PQ = 0.$$

12. Take any point V on the line of the involution. Then $OP = VP - VO = x - r$, say; so $OP' = x' - r$.

$\therefore$ $OP \cdot OP'$ = constant gives $(x-r)(x'-r)$ = constant which is of the form $kxx' + l(x+x') + n = 0$.

Hence *the distances of pairs of points in an involution from any point on the line satisfy the relation* $kxx' + l(x+x') + n = 0$, *where* k, l *and* n *are constants.*

Conversely, *if this relation be satisfied, the pairs of points form an involution.*

For $kxx' + l(x+x') + n = 0$ can be thrown into the form $(x-r)(x'-r)$ = constant; which is the same as

$$OP \cdot OP' = \text{constant}.$$

Or thus. If $(AA', BB', CC', \ldots)$ be in involution, then $(AA'BB'CC', \ldots)$ is homographic with $(A'AB'BC'C, \ldots)$. Hence corresponding points in the two ranges are connected by a relation of the form $kxx' + lx + mx' + n = 0$. Also, as there is no distinction in an involution between P and P', we must have $l = m$. Conversely, if $kxx' + l(x+x') + n = 0$, P and P' generate homographic ranges in which P and P' are interchangeable. Hence P and P' generate an involution.

The double points of the involution defined by

$$kxx' + l(x+x') + n = 0$$

are given by the equation $kx^2 + 2lx + n = 0$; for at a double point $x = x'$. To get the centre we put $x = \infty$ in

$$k + l(1/x + 1/x') + n/xx' = 0;$$

hence $$x' = -l/k.$$

Ex. 1. *Show that* P, P' *determine an involution if*

$$AP \cdot B'P' + \lambda \cdot AP + \mu \cdot B'P' + \nu = 0,$$

provided $\lambda - \mu = AB'$.

Ex. 2. *Show that* P, P' *determine an involution if*

$$2 \cdot AP \cdot BP' = AB \cdot PP';$$

and that A *and* B *are the double points.*

Ex. 3. *Show that* P, P' *determine an involution if*

$$AP + B'P' = c;$$

and that one double point is at infinity.

13. The relation $kxx' + l(x+x') + n = 0$ is called a (1, 1) and symmetrical relation because, when x is given, x' is given uniquely and, when x' is given, x is given uniquely; also the relation is symmetrical in x and x', so that x and x' are given by the same relation. Just as in the case of homography we have—*If the points P and P' move on a line in such a manner that when P is given, then P' is known uniquely by a rational construction, and when P' is given, then P is known uniquely by the same construction, then P and P' generate an involution range, of which they are corresponding points.*

For since P and P' are connected by a rational (1, 1) construction, they are connected by a (1, 1) relation; which is symmetrical in x and x' because P and P' are given by the same construction.

XVI AND XVII

1. A conic is drawn through the common points of two homographic ranges $AB\ldots$, $A'B'\ldots$ on the same line. P is any point on the conic, and PA, PA' cut the conic again at a, a'. Show that a, a' generate homographic ranges on the conic, and that the ranges obtained by varying P are identical.

2. Circles of a coaxal system whose centres are A, B, C touch the sides MN, NL, LM of a triangle at P, Q, R, and circles of the same system whose centres are A', B', C' pass through the vertices L, M, N of the triangle. If P, Q, R are collinear, show that (AA', BB', CC') is an involution.

3. If R bisects CC' and R' corresponds to R in the involution of which C, C' are corresponding points and O the centre, show that $RC^2 = RR'.RO$.

4. E and F are the double points of the involution (AA', BB'). Show that
$$\frac{AB.AB'}{A'B.A'B'} = -\frac{AE.AF}{A'E.A'F}.$$

5. If each of the sides of a triangle meets three circles in pairs of points in involution, the circles are coaxal.

6. The three circles drawn through a given point V, one coaxal with the circles c_1 and c_2, one coaxal with the circles c_2 and c_3, and one coaxal with the circles c_3 and c_1, are coaxal.

7. Prove the following construction for the common points of the two homographic ranges $(ABC\ldots)$ and $(A'B'C'\ldots)$. Take any point P and let the circles PAB' and $PA'B$ cut in Q, and let the circles PAC' and $PA'C$ cut in R; then the circle PQR will cut AA' in the required points.

8. If E is a double point of the involution (AA', BB') and if P bisects AA' and Q bisects BB', show that

$$A'A^2/PE - B'B^2/QE = 4PQ.$$

9. If (AA', BB', CC') is an involution and P any point on AA', show that

$$\text{(i)}\quad \frac{CA}{C'A'}\,.\,BB' + \frac{CB}{C'B'}\,.\,B'A + \frac{CB'}{C'B}\,.\,AB = 0,$$

$$\text{(ii)}\quad \frac{CA}{C'A'}\,.\,BB'\,.\,PA' + \frac{CB}{C'B'}\,.\,B'A\,.\,PB' + \frac{CB'}{C'B}\,.\,AB\,.\,PB = 0.$$

10. If $(PP', QR, Q'R')$ and $(PP', QR', Q'R)$ are involutions, show that P, P' are the double points of the involution determined by (QQ', RR').

CHAPTER XVIII

PENCILS IN INVOLUTION

1. The pencil of lines VA, VA', VB, VB', VC, VC', ... is said to form a pencil in involution if

$$V(AA'BB'CC'...) = V(A'AB'BC'C...).$$

Any transversal cuts an involution pencil in an involution range; and, conversely, the pencil joining any involution range to any point is in involution.

Let a transversal cut an involution pencil in the pairs of points AA', BB', CC', ... Then, since

$$V(AA'BB'CC'...) = V(A'AB'BC'C...),$$

we have

$$\begin{aligned}(AA'BB'CC'...) &= V(AA'BB'CC'...)\\ &= V(A'AB'BC'C...) = (A'AB'B'C'C...).\end{aligned}$$

Hence $\quad (AA'BB'CC'...) = (A'AB'BC'C...).$

Hence $(AA', BB', CC', ...)$ is an involution.

Conversely, if $(AA', BB', CC', ...)$ is an involution, we have $(AA'BB'CC'...) = (A'AB'BC'C...)$. Hence

$$\begin{aligned}V(AA'BB'CC'...) &= (AA'BB'CC'...)\\ &= (A'AB'BC'C...) = V(A'AB'BC'C...).\end{aligned}$$

Hence $V(AA', BB', CC', ...)$ is an involution.

Notice that *the reciprocal of a range in involution is a pencil in involution*; for the reciprocals of the homographic ranges $(AA'BB'CC'...)$ and $(A'AB'BC'C...)$ are the homographic pencils $(aa', bb', cc', ...)$ and $(a'a, b'b, c'c, ...)$.

Ex. 1. *If $V(AA', BB', CC')$ is a pencil in involution, show that* $\sin AVB \,.\, \sin B'VC \,.\, \sin C'VA'$

$$= -\sin A'VB' \,.\, \sin BVC' \,.\, \sin CVA.$$

Consider a section of the pencil.

Ex. 2. *If V be any point on the homographic axis of the two homographic ranges $(ABC...)$ and $(A'B'C'...)$ on different lines; show that $V(AA', BB', CC', ...)$ is an involution pencil.*

Let $X'Y$ be the mates of the point $X(=Y')$ where AB and $A'B'$ meet. Then V is on $X'Y$. Hence

$$V(XX'ABC\ldots) = V(XYABC\ldots)$$
$$= V(X'Y'A'B'C'\ldots) \text{ by homography} = V(X'XA'B'C'\ldots).$$

Hence $V(XX', AA', BB', \ldots)$ is an involution.

Ex. 3. *Reciprocate Ex.* 2.

Ex. 4. *Any two homographic pencils can be placed in two ways so as to be in involution.*

Let the pencils be $V(ABC\ldots) = V'(A'B'C'\ldots)$. First, superpose the pencils so that V is on V' and VA on $V'A'$. This can be done in two ways. Let $VX(=V'X')$ be the other common line of the two pencils

$$V(ABC\ldots) = V(AB'C'\ldots).$$

Then in the original figure $AVX = A'V'X'$. Second, place V on V' and VA on $V'X'$ and VX on $V'A'$. The two pencils are now in involution; for $VA(=V'X')$ has the same mate, viz. $V'A'(=VX)$ whichever pencil it is supposed to belong to.

If the vertices are at infinity, place the pencils so that all the rays are parallel. Let any line now cut them in the homographic ranges $(abc\ldots) = (a'b'c'\ldots)$. Now slide $(a'b'c'\ldots)$ along $(abc\ldots)$ until the two ranges are in involution (either by Ex. 7 of XVII. 6, or by a construction similar to the above).

2. *A pencil of rays in involution has two double rays* (*i.e. rays each of which coincides with its corresponding ray*), *and the double rays divide harmonically the angle between every pair of rays.*

Let any transversal cut the pencil in the involution $(AA', BB', CC', \ldots)$, and let E, F be the double points of this involution. Then VE corresponds to itself in the involution $V(AA', BB', CC', \ldots)$ since E corresponds to itself in the involution $(AA', BB', CC', \ldots)$. Hence VE is a double ray. So VF is a double ray. Also $(AEA'F)$ is a harmonic range; hence $V(AEA'F)$ is a harmonic pencil. Similarly VE, VF divide each of the angles BVB', CVC', ... harmonically.

Note that there is nothing in an involution pencil which is analogous to the centre of an involution range. In fact

the point at infinity in the range AA', BB', CC', ... will project into a finite point on another transversal, and O will project into the mate of this finite point.

If, however, V is at infinity, i.e. if the rays of the pencil are parallel, then all sections of the pencil are similar, and there is a central ray which is the locus of the centres of all the involution ranges determined on transversals.

If the angles AVA', BVB', CVC', ... be divided harmonically by the same pair of lines, the pencil $V(AA', BB', CC', ...)$ is in involution.

For, in a section, AA', BB', ... are harmonic with EF.

If the double rays of an involution pencil are real, none of the angles bounded by corresponding rays overlap, and if the double rays are imaginary, all the angles overlap.

For a similar theorem holds for the double points of a section of the pencil.

Ex. 1. *If the angles AVA', BVB', ... are bisected by the same line, then $V(AA', BB', ...)$ is an involution.*

Ex. 2. *If the double rays of a pencil in involution be perpendicular, they bisect all the angles bounded by corresponding rays.*

Ex. 3. *If two angles AVA', BVB' bounded by corresponding rays of a pencil in involution have the same bisectors, then all such angles have the same bisectors.*

Ex. 4. *Find the locus of a point at which every segment (AB) of an involution subtends the same angle as the corresponding segment ($A'B'$).*

The circle on EF as diameter.

Ex. 5. *Through any point O are drawn chords AA', BB', CC', ... of a conic; show that AA', BB', CC' subtend an involution pencil at any point of the polar of O.*

Ex. 6. *Reciprocate Ex.* 5.

Ex. 7. *If $ABA'B'$ be four points on a conic, and if through any point O on the external side UW of the harmonic triangle of $ABA'B'$ there be drawn two tangents OT and OT' to the conic; show that $O(AA', BB', TT')$ is a pencil in involution, the double lines being OU and OV.*

Ex. 8. *Show that in an involution pencil*

$$\frac{\sin AVB \,.\, \sin AVB'}{\sin A'VB \,.\, \sin A'VB'} = \frac{\sin AVC \,.\, \sin AVC'}{\sin A'VC \,.\, \sin A'VC'} = \frac{\sin^2 AVE}{\sin^2 A'VE}.$$

3. *If AVA', BVB', CVC', ... be all right angles, then the pencil $V(AA', BB', CC', ...)$ is in involution.*

We have to show that

$$V(AA'BB'CC'...) = V(A'AB'BC'C...).$$

Produce AV to a, BV to b, and so on.

Then if we place VA on VA', VA' will fall on Va, and so on. Hence the two pencils

$$V(AA'BB'...) \text{ and } V(A'aB'b...)$$

are superposable and therefore homographic. But

$$V(A'aB'b...)$$

is the same as $V(A'AB'B...)$; hence $V(AA'BB'...)$ and $V(A'AB'B...)$ are homographic.

Otherwise:—From the vertex V drop the perpendicular VO on any transversal $AA'BB'...$. Then, since AVA' is a right angle, we have $VO^2 = AO \,.\, OA'$.

Hence $OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = ...$

Hence $(AA', BB', CC', ...)$ is an involution range.

Hence $V(AA', BB', CC', ...)$ is an involution pencil.

Ex. *If through the centre of an overlapping involution $(AA', BB', ...)$, there be drawn VO perpendicular to AA' and such that $VO^2 = AO \,.\, OA'$, then any four points of the involution subtend at V a pencil superposable to that subtended by their mates.*

4. *In every involution pencil, there is one pair of corresponding rays which is orthogonal; and if more than one pair be orthogonal, then every pair is orthogonal.* (See also XX. 6.)

Take any transversal cutting the pencil in the involution $(AA', BB', CC', ...)$. Through the vertex V draw the circles VAA', VBB' cutting again in V'. Let VV' cut AA' in O. Then $OA \,.\, OA' = OV \,.\, OV' = OB \,.\, OB'$. Hence O is the centre of the involution. Hence

$$OC \,.\, OC' = OA \,.\, OA' = OV \,.\, OV'.$$

Hence the four points V, V', C, C' are concyclic; i.e. the

circle VCC' passes through V'. Similarly all such circles pass through V'. Also every circle through VV' cuts AA' in a pair of points PP' of the involution; for

$$OP \,.\, OP' = OV \,.\, OV' = OA \,.\, OA'.$$

Let the line bisecting VV' at right angles cut AA' in Q. Describe a circle with Q as centre and with QV as radius, cutting AA' in PP'. This circle will pass through V' and hence P, P' are a pair in the involution. Also PVP' is a right angle.

This construction fails in only one case, viz. when VV' is perpendicular to AA'. In this case, the orthogonal pair are VV' and the perpendicular to VV' through V. For the point at infinity on AA' corresponds to O which is on VV'.

Also if two pairs are orthogonal, every pair is orthogonal. For suppose AVA', BVB' are right angles. Then the centres of the circles AVA' and BVB' are on AA'. Hence AA' bisects VV' orthogonally. Hence the centres of all the circles AVA', BVB', CVC', ... are on AA'. Hence all the angles AVA', BVB', CVC', ... are right angles.

Notice that we have incidentally proved the following construction for the centre O and the point corresponding to a given point P in the involution determined by the pairs of points AA', BB', viz. Take any point V and let the circles AVA' and BVB' intersect again at V'; then VV' cuts AA' at the centre O, and the circle PVV' cuts AA' again at the point P' corresponding to P.

Again if with centre O we draw a circle with radius equal to a tangent OT from O to the circle AVA', this circle cuts AA' in the double points E, F; for

$$OE^2 = OF^2 = OT^2 = OA \,.\, OA'.$$

Ex. 1. *Show that a given line VX through the vertex always bisects one of the angles AVA', BVB', ... of an involution.*

Take AA' perpendicular to VX, and take the centre of the circle VPP' on VX.

Ex. 2. *Given two pairs A, A' and B, B' of an involution and the middle point of CC', construct C and C'.*

Ex. 3. *Given a segment AA' of an involution and the centre O, construct the mate of C.*

Ex. 4. *Given two involutions (AA', BB', ...) and (aa', bb', ...) on the same line: find two points which correspond to one another in both involutions.*

Let aa', bb', ... give v', just as AA', BB', ... gave V'. Now draw the circle $VV'v'$.

Ex. 5. *If VR is one of the orthogonal rays of the involution pencil $V(AA', BB', CC', ...)$, show that* $\tan RVA \,.\, \tan RVA'$ *is constant.*

Take a section perpendicular to VR.

5. *Every overlapping pencil in involution can be projected into an orthogonal involution.*

Let any transversal cut the pencil in the overlapping involution of points (AA', BB', CC', ...). On AA', BB' as diameters describe circles. Since AA', BB' overlap, the circles will cut in two real points U and U'. Now, since in the pencil in involution $U(AA', BB', CC', ...)$ two pairs of rays, viz. UA, UA' and UB, UB', are orthogonal, it follows that every pair is orthogonal.

Rotate U about AA' out of the plane of the paper. With any vertex W on the line joining U to the vertex V of the given pencil, project the given pencil on to any plane parallel to the plane UAA'. Then VA projects into a line parallel to UA, and VA' projects into a line parallel to UA'; hence AVA' projects into a right angle; similarly BVB', CVC', ... project into right angles.

Ex. 1. (AA', BB', CC', ...) *is an involution. Show that the circles on AA', BB', CC', ... as diameters are coaxal.*

Ex. 2. *Show also that AA', BB', CC', ... subtend right angles at two points in the plane. When are these points real?*

***6.** *If P, Q, R be the fourth harmonics of the point X for the segments AA', BB', CC' of an involution range, then*

$$\frac{PA^2}{XA^2}\cdot\frac{QR}{XP}+\frac{QB^2}{XB^2}\cdot\frac{RP}{XQ}+\frac{RC^2}{XC^2}\cdot\frac{PQ}{XR}+\frac{PQ\,.\,QR\,.\,RP}{XP\,.\,XQ\,.\,XR}=0.$$

Join the points to any vertex V; and cut the pencil so formed by a transversal aa', bb', cc', p, q, r, x. Then, since

the given relation is homogeneous in the points, we see that the relation need only be proved of the projections aa', &c. of the given points. Now take aa' parallel to VX. Then x is at infinity. Hence

$$\frac{xa^2 \, . \, xp}{xb^2 \, . \, xq} = 1, \quad \frac{xa^2 \, . \, xp}{xc^2 \, . \, xr} = 1, \quad \frac{xa^2 \, . \, xp}{xp \, . \, xq \, . \, xr} = 1.$$

Hence the given relation is true if

$$pa^2 \, . \, qr + qb^2 \, . \, rp + rc^2 \, . \, pq + pq \, . \, qr \, . \, rp = 0.$$

But now p, q, r bisect aa', bb', cc'; hence this relation is true by XVII. 11.

Ex. 1. *Show also that*

$$\frac{YA \, . \, YA'}{XA \, . \, XA'} \, . \, QR \, . \, XP + \frac{YB \, . \, YB'}{XB \, . \, XB'} \, . \, RP \, . \, XQ + \frac{YC \, . \, YC'}{XC \, . \, XC'} \, . \, PQ \, . \, XR = 0,$$

where Y *is any point on the same line.*

Ex. 2. *Also*

$$\frac{QR \, . \, XP}{XA \, . \, XA'} + \frac{RP \, . \, XQ}{XB \, . \, XB'} + \frac{PQ \, . \, XR}{XC \, . \, XC'} = 0.$$

Take Y at infinity.

Ex. 3. *Also*

$\dfrac{QR}{XP_1} + \dfrac{RP}{XQ_1} + \dfrac{PQ}{XR_1} = 0$ *if* P_1, Q_1, R_1 *bisect* AA', BB', CC'.

For, by II. 2, end, $XP \, . \, XP_1 = XA \, . \, XA'$, &c.

Ex. 4. *If* $(AA', BB') = (XC, AA') = (XC', BB') = -1$, *then* (AA', BB', CC') *are in involution.*

Project A to infinity, and prove that $b^2 = cc'$.

Ex. 5. *If* (AA', BB') *and* (AA', QQ') *be harmonic, then*

$$\frac{PA \, . \, PA'}{QA \, . \, QA'} \, . \, BB' \, . \, QQ' + \frac{PB^2}{QB} \, . \, B'Q' + \frac{B'P^2}{QB'} \, . \, Q'B = 0.$$

7. Just as in the case of homographic pencils, we see, by taking a section of the pencil, that—*If* VP *and* VP' *are rays such that if* VP *is given, then* VP' *is known uniquely by a rational construction and when* VP' *is given, then* VP *is known uniquely by the same construction, then* VP *and* VP' *generate an involution pencil of which they are corresponding rays.*

Ex. 1. *If VX be any line, show that the* (1, 1) *symmetrical relation is of the form*

$$k \tan XVP . \tan XVP' + l . \tan XVP + l . \tan XVP' + n = 0,$$

where k, l and n are constants.

Ex. 2. *If VX and VY be fixed lines, and VP, VP' be variable lines satisfying the condition*

$$\sin XVP \div \sin YVP = -\sin XVP' \div \sin YVP',$$

then VP, VP' generate an involution.

For in the section X, Y are harmonic with P, P'.

XVIII

1. Two homographic pencils have their vertices at infinity. Show that any line through their homographic pole determines an involution of which the pole is the centre.

2. If E, F are the limiting points of the circles on the collinear segments AA', BB' as diameters, show that the circles on AB, $A'B'$, EF as diameters are coaxal.

3. If

$$(CP, AA') = (CQ, BB') = (PP', BB') \\ = (QQ', AA') = (CC', P'Q') = -1,$$

show that AA', BB', CC' are in involution.

4. If (AA', BB', CC', DD') is an involution, and if $(LP, AA') = (LQ, BB') = (LR, CC') = (LS, DD') = -1$, show that $(PQ, RS) = (AB, CD) \times (A'B', CD)$.

5. If VA', VB', VC' are three bisectors of the angles BVC, CVA, AVB (either three external, or one external and two internal), then $V(AA', BB', CC')$ is an involution.

6. If VE and ve are any two double rays of the involution pencils $V(AA', BB', \ldots)$ and $v(aa', bb', \ldots)$, and if VF and vf are the other double rays, show that if VE and ve intersect at G and VF and vf at H, then the pencils are in perspective with GH as axis of perspective; i.e. if VA and va meet on GH, so do VA' and va'.

CHAPTER XIX

INVOLUTION OF CONJUGATE POINTS AND LINES

1. *The pairs of points on a line which are conjugate for a conic form an involution.*

Let l be the line and L its pole. Let AA', BB', CC', ... be the pairs of conjugate points on l. Then the polar of A which lies on l passes through L. Also the polar of A by hypothesis passes through A'. Hence LA' is the polar of A. So LA is the polar of A', and so on. Hence $(AA'BB'CC' \ldots)$ of poles $= L\,(A'AB'BC'C \ldots)$ of polars $= (A'AB'BC'C \ldots)$. Hence $(AA', BB', CC', \ldots)$ form an involution.

The double points of the involution of conjugate points on a line are the meets of the line and the conic.

For these meets are harmonic with every pair of conjugate points on the line.

This affords another proof that conjugate points on a line generate an involution.

2. *The pairs of lines through a point which are conjugate for a conic form an involution.*

Let L be the point and l its polar. Let LA, LA'; LB, LB'; LC, LC'; ... be the pairs of conjugate lines, the points AA', BB', CC', ... being on l. Then the pole of LA which passes through L is on l. But the pole of LA by hypothesis lies on LA'. Hence the pole of LA is A'; so the pole of LA' is A, and so on. Hence $L\,(AA'BB'CC' \ldots)$ of polars $= (A'AB'BC'C \ldots)$ of poles $= L\,(A'AB'BC'C \ldots)$. Hence $L\,(AA', BB', CC', \ldots)$ form an involution.

The double lines of the involution of conjugate lines through a point are the tangents from the point.

For these tangents are harmonic with every pair of conjugate lines through the point.

This affords another proof that conjugate lines through a point generate an involution.

Ex. 1. *Through every point can be drawn a pair of lines which are conjugate for a given conic and also perpendicular.*

Ex. 2. *If two pairs of conjugate lines at a point are perpendicular, all pairs are perpendicular.*

Ex. 3. *Given that l is the polar of L, and given that ABC is a self-conjugate triangle, construct the tangents from L.*

The pole of LA is the intersection of l and BC.

3. An important case of conjugate lines is that of conjugate diameters, i.e. conjugate lines at the centre. The double lines of the involution of conjugate diameters are the tangents from the centre, i.e. the asymptotes.

Ex. 1. *Conjugate diameters of a hyperbola do not overlap, and conjugate diameters of an ellipse do overlap.*

Ex. 2. *The conjugate diameters CQ, CE cut the tangent at P in R, R'; show that $RP \cdot PR' = CD^2$.*

For P is the centre of the involution determined by the variable conjugate diameters CQ, CE on the tangent at P. Also in the hyperbola the double points are on the asymptotes. Hence $RP \cdot PR' = -PT^2 = CD^2$. In the ellipse the diagonals of the quadrilateral of tangents at P, P', D, D' give a case of CQ, CE. Hence $RP \cdot PR' = CD^2$.

Ex. 3. *Parallel tangents of a conic cut the tangent at P in R, R'; show that $RP \cdot PR' = CD^2$.*

For, on completing the parallelogram of tangents, we see that CR and CR' are conjugate diameters.

Ex. 4. *The conjugate diameters CQ, CE cut the tangents at the ends of the diameter PP' in R, R' respectively; show that*

$$PR \cdot P'R' = CD^2.$$

Reflect R' in the centre.

4. Defining the *principal axes* of a central conic as a pair of conjugate diameters which are at right angles, we can prove that—

The principal axes of a conic are always real.

For by XVIII. 4, one real pair of rays of an involution pencil is always orthogonal.

A central conic (unless it be a circle) has only one pair of principal axes.

For by XVIII. 4, if two pairs of rays of the involution pencil are orthogonal, then every pair is orthogonal, i. e. the conic is a circle.

Ex. *Given the centre of a conic and a self-conjugate triangle, construct the axes and asymptotes.*

Let O be the centre and ABC the triangle. Through O draw OA', OB', OC' parallel to BC, CA, AB. Then the pole of OA is the point at infinity on BC. Hence OA and OA' are conjugate diameters; so OB, OB' and OC, OC'. Hence, the asymptotes are the double lines, and the axes are the orthogonal pair of the involution $O(AA', BB', CC')$.

***5.** *The feet of the normals which can be drawn from any point to a central conic are the meets of the given conic, and a certain rectangular hyperbola which has its asymptotes parallel to the axes of the given conic, and which passes through the centre of the given conic, and through the given point.*

Let O be the given point. Take any diameter CP, and let the perpendicular OY on CP cut the conjugate diameter CD in Q. Then, taking several positions of P, &c.,

$$\begin{aligned} C(Q_1Q_2 \ldots) &= C(D_1D_2 \ldots) = C(P_1P_2 \ldots) \\ &= C(Y_1Y_2 \ldots) = O(Y_1Y_2 \ldots) = O(Q_1Q_2 \ldots). \end{aligned}$$

Hence the locus of Q is a conic through C and O.

This conic is a rectangular hyperbola with its asymptotes parallel to the axes, as we see by making CP coincide with CA and CB in succession. Now let R be the foot of a normal from O to the given conic, then R is on the above rectangular hyperbola; for, drawing CP perpendicular to OR, OY meets CD, i. e. CR, in R.

Ex. 1. *The same conic is the locus of points Q such that the perpendicular from Q on the polar of Q passes through O.*

For QO, being perpendicular to the polar, is perpendicular to the diameter conjugate to CQ.

Ex. 2. *The normals at the four points where a conic is cut by a rectangular hyperbola which passes through the centre and has its asymptotes parallel to the axes, are concurrent at a point on the rectangular hyperbola.*

For let the normal at one of the meets R cut the hyperbola again in O.

Ex. 3. *Eight lines can be drawn from a given point to meet a given central conic at a given angle.*

Ex. 4. *Deduce the corresponding theorems in the case of a parabola.*

Here the centre Ω is on the curve; hence one of the meets is the point Ω. Rejecting this, we see that three normals and six obliques can be drawn from any point to a parabola.

Ex. 5. *If OL, OM, ON, OR be concurrent normals to a conic, the tangents at L, M, N, R touch a parabola which also touches the axes of the conic and the polar of O for the conic.*

Reciprocate for the given conic.

6. A *common chord* of two conics is the line joining two common points of the conics.

On a common chord of two conics the involution of conjugate points is the same for each conic, the double points being the common points.

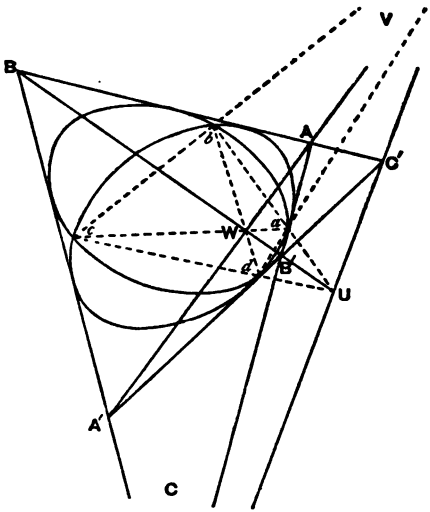
Conversely, *if two conics have on a line the same involution of conjugate points, this line is a common chord, the double points of the involution being the common points of the two conics.*

Two common chords of two conics which do not cut on the conics may be called *a pair of common chords.* We know that a pair of common chords meet in a vertex of a common self-conjugate triangle of the two conics. Conversely, *every point which has the same polar for two conics is the meet of a pair of common chords.*

Let V be the point. Join V to any common point a of the two conics. Let Va cut the polar of V in P and the conics again in d and d'. Then (VP, ad) is harmonic, and also (VP, ad'). Hence d and d' coincide, i. e. Va passes through a second common point. So Vb passes through c.

Hence *two conics have only one common self-conjugate triangle;* for if $U'V'W'$ be a self-conjugate triangle, and UVW the harmonic triangle belonging to the meets of the conics,

then U' coincides with U, V, or W, and so on. (See also XXV. 12.)



If, however, the *two conics have double contact* the above proof breaks down, and *there is an infinite number of common self-conjugate triangles.* (See XI. 7.)

Ex. 1. *The common chords which pass through one of the vertices of the common self-conjugate triangle of two conics are in involution with the pairs of tangents from this point.*

UV, UW being the double lines.

Ex. 2. *Reciprocate Ex.* 1.

7. A *common apex* of two conics is the meet of two common tangents of the conics.

At a common apex of two conics the involution of conjugate lines is the same for each conic, the double lines being the common tangents.

Conversely, *if two conics have at a point the same involution of conjugate lines, the point is a common apex, the double lines of the involution being the common tangents of the two conics.*

The join of a pair of common apexes of two conics has the same pole for both conics.

Conversely, *if a line have the same pole for two conics, this line is the join of a pair of common apexes of the conics.*

These results follow by Reciprocation from § 6.

8. Since two conics have only one common self-conjugate triangle, it follows that *the harmonic triangle of the quadrangle of common points coincides with the harmonic triangle of the quadrilateral of common tangents.*

Hence, if UVW be the harmonic triangle of the quadrangle formed by the common points a, b, c, d, and if AA', BB', CC' be the opposite vertices of the quadrilateral formed by the common tangents of the two conics, then AA' being a side of the common self-conjugate triangle, must coincide with UV, VW, or WU, say with VW, as in the figure. So BB' coincides with WU, and CC' with UV.

The polars of any common apex of two conics for the two conics pass through the meet of two common chords of the conics.

Take the common apex B. Now B is on WU, the polar of V. Hence the polar of B for either conic passes through V, the meet of the common chords ad, bc.

The common chords ad, bc are said to *belong* to the common apex B. So to every common apex belong two common chords.

Similarly, *the poles of any common chord of two conics for the two conics lie on the join of two common apexes of the conics;* and these apexes are said to *belong* to the chord.

Homothetic figures.

***9.** Given any figure of points $P, Q, R, \ldots$, and any point O (called *the centre of similitude*), and any ratio λ (called *the ratio of similitude*), we can generate another figure of points $P', Q', R', \ldots$ thus—In OP take the point P', which is such that $OP' = \lambda . OP$, and similarly construct $Q', R', \ldots$. The figures $PQR\ldots$ and $P'Q'R'\ldots$ are called similar and similarly situated figures, or *homothetic figures.*

The following properties of homothetic figures follow from the definition by elementary geometry—

Corresponding sides of the two figures (e. g. PQ and $P'Q'$) are parallel and in the ratio λ (i. e. $P'Q' = \lambda \,.\, PQ$).

Corresponding angles of the two figures are equal

$$(e.g.\ \angle PQR = \angle P'Q'R').$$

Ex. *The triangles ABC, $A'B'C'$ are coaxal. P, Q, R are three points on the axis. Show that if AP, BQ, CR concur, so do $A'P$, $B'Q$, $C'R$.*

Project the axis to infinity.

***10.** *If two conics are homothetic, the diameters conjugate to parallel diameters are themselves parallel.*

Consider the point corresponding to the centre of the first conic; it will be a point in the second conic, all chords through which are bisected at the point, i. e. it will be the centre of the second conic. Take any pair of conjugate diameters PCP' and DCD' of the first conic; and let pcp' be the diameter of the second conic parallel to PCP'. Then, corresponding to DCD' in the first conic, we shall have dcd' in the second conic which bisects chords parallel to pcp', i. e. dcd' is the diameter conjugate to pcp'. Hence, to a pair of conjugate diameters of the first conic correspond a parallel pair of conjugate diameters of the second conic.

***11.** *Two conics will be homothetic, if two pairs of conjugate diameters of the one are parallel to two pairs of conjugate diameters of the other.*

For then, by considering the involutions of conjugate diameters, we see that every pair is parallel to some pair. Take any diameter PCP' of the first conic, and through P and P' draw lines parallel to a pair of conjugate diameters; these lines meet in a point Q on the conic. Let pcp' be the diameter in the second conic parallel to PCP', and through p and p' draw lines parallel to PQ and $P'Q$. These will meet in a point q on the second conic; for they are parallel to a pair of conjugate diameters of the first conic, and therefore parallel to a pair of conjugate diameters of the second conic. And clearly the points Q and q generate homothetic figures, the centre of similitude being the intersection of Pp and Cc.

Homothetic conics are conics which meet the line at infinity in the same points.

If the conics are homothetic, their conjugate diameters are parallel. Hence the asymptotes, being the double lines of the involutions of conjugate diameters, are parallel, i.e. meet the line at infinity in the same points. And both conics pass through these points.

Conversely, if two conics pass through the same two points at infinity, they are homothetic. For since the conics pass through the same two points at infinity, the asymptotes of the two conics are parallel. Hence the conjugate diameters, being harmonic with the asymptotes, are parallel. Hence the conics are homothetic.

Ex. 1. *Through three given points, draw a conic homothetic to a given conic.*

To draw through ABC a conic homothetic to a, through the middle point of AB draw a line parallel to the diameter bisecting chords of a parallel to AB. This line passes through the centre O of the required conic. Similarly BC gives us another line through O. Hence the centre of the required conic and three points upon it are known.

Ex. 2. *Touching three given lines, draw a conic homothetic to a given conic.*

Draw tangents of the conic parallel to the sides of the given triangle. It will be found that we thus have four triangles homothetic to the given triangle. Taking any one of these triangles, and dividing the sides of the given triangle similarly, we get the points of contact of a homothetic conic.

CHAPTER XX

INVOLUTION RANGE ON A CONIC

1. THE pairs of points AA', BB', CC', ... on a conic are said to form an *involution range on a conic*, or briefly, to be in involution when the pencil $V(AA', BB', CC', ...)$ subtended by them at a point V on the conic is in involution.

An involution range on a conic has two double points, which form with any pair of points of the involution, two pairs of harmonic points on the conic.

For if VX, VY are the double lines of the involution pencil $V(AA', BB', CC', ...)$, then $V(XY, AA')$ is harmonic, i. e. (XY, AA') is harmonic.

2. *If the pairs of points AA', BB', CC', ... on a conic be in involution, then the chords AA', BB', CC', ... are concurrent; and conversely, if the chords AA', BB', CC', ... of a conic be concurrent, then the pairs of points AA', BB', CC', ... on the conic are in involution.*

If $(AA', BB', CC', ...)$ form an involution on the conic, we have $(AA'BB'CC' ...) = (A'AB'BC'C ...)$. Hence

$$A(AA'BB'CC' ...) = A'(A'AB'BC'C ...).$$

These pencils have the common corresponding ray AA' and $A'A$. Hence they are in perspective, i.e. the intersections

$$(AB;\ A'B'),\ (AB';\ A'B),\ (AC;\ A'C'),\ (AC';\ A'C), \cdots$$

all lie on a fixed line. But, by the quadrangle construction for the polar of a point, AA', BB' meet at the pole of the line joining $(AB;\ A'B')$ to $(AB';\ A'B)$; hence AA', BB' pass through a fixed point, viz. the pole of the above fixed line. Similarly CC', ... all pass through this point.

Again, if the chords AA', BB', CC', ... are concurrent

at O, the pairs of points AA', BB', CC', ... on the conic are in involution. It is sufficient to prove that

$$A(AA'BB'CC'...) = A'(A'AB'BC'C...),$$

i.e. that $(AA; A'A')$, $(AB; A'B')$, $(AB'; A'B)$, ... all lie on a fixed line: and this is true; for AA and $A'A'$ (viz. the tangents at A and A') and also $(AB; A'B')$, $(AB'; A'B)$, ... all lie on the polar of O.

The point of concurrence O is called the *pole of the involution*; and the line on which $(AB; A'B')$, $(AB'; A'B)$, ... lie, is called *the axis of the involution*; and we have proved that the *pole of the involution is the pole of the axis of the involution for the conic.*

We may also prove that *chords of a conic through a fixed point cut the conic in pairs of points in involution* by (1, 1) correspondence. For let any line through O cut the conic at P and P'; and take any point V on the conic. Then when VP is given, P is known, OP is known, P' is known, and VP' is known, all uniquely. So if VP' is known, VP is known uniquely and by the same construction—which is clearly rational. Hence VP and VP' generate an involution pencil; i.e. P and P' generate an involution on the conic.

Conversely, *if $(AA', BB', CC', ...)$ is a range in involution on the conic, then AA', BB', CC', ... are concurrent.* For let AA', BB' meet at O and let OC cut the conic again at C''. Then by hypothesis (AA', BB', CC') is an involution; and by the first part (AA', BB', CC'') is an involution. Hence C' and C'' coincide; i.e. CC' passes through O; so all such lines pass through O.

For a proof by projection see XXIX. 4, Ex. 10.

An interesting particular case of the theorem of this article is—*If the lines joining a point O to any number of points A, B, C, ... on a conic cut the conic again at A', B', C', ... then* $(ABC...) = (A'B'C'...)$.

Take any point V on the conic. Then since

$$AA', BB', CC', ...$$

are concurrent at O, $(AA', BB', CC', ...)$ is an involution, i.e.

$V(AA'BB'CC'\ldots) = V(A'AB'BC'C\ldots)$. Hence, selecting the first, third, fifth, ... letters on each side, we have

$$V(ABC\ldots) = V(A'B'C'\ldots)$$

which is the same statement as $(ABC\ldots) = (A'B'C'\ldots)$.

The reader should be careful to notice that $(ABC\ldots)$ is not equal to $O(ABC\ldots)$ in the above proof; for $(ABC\ldots)$ is only equal to $V(ABC\ldots)$ if V is on the conic.

The double points of the involution are the points where the axis of the involution cuts the conic; for when A and A' coincide, AA' becomes the tangent at A, and hence A lies on the polar of O, since AA' passes through O.

Hence, the double points are real if the pole of the involution is outside the conic, and imaginary if the pole is inside the conic.

Notice that the axis of the involution $(AA', BB', CC', \ldots)$ on the conic is the axis of homography of the homographic ranges $(AA'BB'CC'\ldots)$ and $(A'AB'BC'C\ldots)$ on the conic; which gives another proof that the axis meets the conic in the double points.

If O is on the conic, we get the curious case of an involution on the conic in which all the points $A, B, C, \ldots$ and the double points coincide at O, whilst $A', B', C', \ldots$ may be anywhere on the conic.

3. *If two segments bounded by corresponding points (such as AA', BB') of an involution on a conic overlap, every two of such segments overlap, and the double points are imaginary, i. e. the pole of the involution is inside the conic. So in a non-overlapping involution on a conic, the double points are real and the pole is outside.*

For consider the pencil subtended at any point on the conic by the points in involution.

Ex. 1. *A set of parallel lines cuts a conic in pairs of points in involution.*

Ex. 2. *Two chords AA', BB' of a conic cut in U, and OT is the tangent at any point O on the conic; show that $O(AA', BB', TU)$ is an involution.*

Consider the involution of U.

Ex. 3. *Through a given point O draw the chord XX' of a conic, such that $(AA', XX') = (BB', XX')$ where A, A', B, B' are any four points on the conic.*

Join O to the meet of AB' and $A'B$.

Ex. 4. *DE is a fixed diameter of a conic. PQ is a variable chord of the conic. The tangent at E meets DP in A and DQ in B. If A, B generate an involution, PQ passes through a fixed point. If $EA \cdot EB$ be constant, the fixed point lies on DE. If $1/EA + 1/EB$ be constant, the fixed point lies on EA.*

In the first case E corresponds to the point at infinity in the involution on EA, and hence DE is one position of PQ; in the second case E is a double point, and hence EE is a position of PQ.

Ex. 5. *From a fixed point O perpendiculars are drawn to the pairs of lines of a pencil in involution, meeting them in AA', BB', ...; show that the lines AA', BB', ... are concurrent.*

Consider the circle on OV as diameter.

Ex. 6. *Through fixed points U, V are drawn the variable chords RP and RQ of a conic; show that P and Q generate homographic ranges on the conic, and that the common points lie on the line UV.*

Ex. 7. *Through a fixed point O is drawn the variable chord PP' of a conic. A and B are fixed points on the conic. PB, $P'A$ meet in Q, and PA, $P'B$ meet in R. Show that Q and R move on the same fixed conic.*

$$\text{For } A\,(QR) = A\,(P'P) = B\,(PP') = B\,(QR).$$

Ex. 8. *Through a centre of similitude of two circles are drawn four lines meeting one circle in $ABCD$, $A'B'C'D'$, and the other circle in $abcd$, $a'b'c'd'$. Show that*

$$(ABCD) = (A'B'C'D') = (abcd) = (a'b'c'd').$$

For $(ABCD) = (A'B'C'D')$ by the involution of $O = (a'b'c'd')$ by similar pencils $= (abcd)$ by the involution of O.

Ex. 9. *A range on a circle and its inverse are homographic.*

For, in the solution of Ex. 8, the ranges $(ABCD)$ and $(a'b'c'd')$ are inverse.

Ex. 10. *A range in involution, whether on a circle or a line, inverts into a range in involution, on a circle or a line.*

Ex. 11. *A variable circle cuts two given circles orthogonally; show that it determines on each circle a range in involution.*

Invert for a meet of the given circles.

Ex. 12. *The pole of the involution $(AA'BB' \ldots)$ on a conic is the same as the homographic pole of the pencils subtended by $AA'BB' \ldots$ and $A'AB'B \ldots$ at any two points on the conic.*

For AA' is one of the cross joins.

4. Reciprocally, *a set of pairs of tangents to a conic are said to be in involution when they cut any tangent to the conic in pairs of points in involution.*

Again, the meets of corresponding tangents which are in involution lie on a line; and conversely, pairs of tangents from points on a line form an involution of tangents.

The double lines of the involution of tangents are the tangents at the meets of the above line with the conic.

These propositions follow at once by Reciprocation.

Notice that *if a set of pairs of tangents be in involution, the set of pairs of points of contact is in involution, and conversely.*

For points on a conic are homographic with the tangents thereat.

Ex. 1. *The pairs of tangents drawn to a parabola from points on a line are parallels to the rays of an involution pencil.*

Ex. 2. *On a fixed tangent of a conic are taken two fixed points A, B, and also two variable points Q, R, such that*

$$(AB, QR) = -1;$$

find the locus of the meet of the other tangents from Q and R.

Ex. 3. *A variable tangent to a conic cuts two fixed lines in A, A'. Show that the points of contact a, a' of the other tangents from A, A' generate homographic ranges on the conic.*

Let AA' touch in a''. Then the ranges a and a'' are in involution, and the ranges a'' and a'. Hence

$$(a \ldots) = (a'' \ldots) = (a' \ldots).$$

Ex. 4. *The fixed tangent OA of a conic meets a variable tangent in X, and the fixed tangent OB meets the parallel tangent in Y. Show that $OX \cdot OY$ is constant.*

Let the parallel tangent meet OA in X'. Then (X, X') generate an involution, since the parallel tangents meet on the line at infinity. Hence $(X) = (X') = (Y)$. And O is the vanishing point of both ranges; for when YX' coincides with OA, X is at infinity and Y at O; also when X is at O, Y is at infinity.

Ex. 5. *AA' is a fixed diameter of a conic; on a fixed line through A' is taken a variable point P, and the tangents from P meet the tangent at A in Q, Q'. Show that $AQ+AQ'$ is constant.*

Q, Q' generate an involution, of which one double point (corresponding to P being at A') is at infinity. Hence the other double point O bisects QQ'. Hence

$$AQ+AQ' = 2AO.$$

Ex. 6. *If P lie on a chord through A instead of A', then $1/AQ+1/AQ'$ is constant.*

5. *Given two involution ranges on a conic or on a line, or two involution pencils at a point; find the pair of points or lines belonging to both involutions.*

The line joining the two poles O_1, O_2 of the involutions on the conic evidently cuts the conic in the required pair of points.

If the ranges are on a line, project the ranges on to a conic by joining to a point on the conic, and project back on to the line the common points on the conic.

In the case of two involution pencils at a point, consider the involutions determined on any conic through the common vertex.

If either of the pairs of double points (or lines) of the given involutions be imaginary, the common pair of points are real; and they are also real when both pairs of double points are real and do not overlap.

For if the involution on the conic, of which O_1 is the pole, has imaginary double points, O_1 is inside the conic; hence $O_1 O_2$ cuts the conic whether O_2 is inside or outside the conic.

Also, if the double points are real and do not overlap, the points sought, being harmonic with both pairs of double points, are the double points of the non-overlapping involution on the conic determined by these double points, and are therefore real.

The cases of involution on the same line or at the same point may be discussed as above.

6. *In a pencil in involution, one pair of rays is always orthogonal; and if two pairs of rays are orthogonal, then every pair is orthogonal.*

Let the rays of the involution pencil $V(AA'BB'CC'\ldots)$ cut a circle through V in the points AA', BB', CC', ... Then AA', BB', CC', ..., being chords joining pairs of points in involution on the circle, meet in a point O. Take K, the centre of the circle, and let OK cut the circle again in ZZ'.

Then VZ, VZ' is an orthogonal pair in the involution pencil. For, since ZZ' passes through K, ZVZ' is a right angle. And since ZZ' passes through O, Z, Z' belong to the involution $(AA', BB', \ldots)$, i. e. VZ, VZ' belong to the given involution pencil $V(AA', BB', \ldots)$.

Again, if two pairs of orthogonal rays exist, viz. VX, VX' and VY, VY', since XX' and YY' both pass through K, we see that O coincides with K. Hence AA', BB', ... all pass through K. Hence all the angles AVA', BVB', ... are right angles.

7. *Chords of a conic which subtend a right angle at a fixed ·t on the conic meet in a point on the normal at the fixed ·t.*

·et the chords QQ', RR', ... of a conic subtend right angles the point P on the conic. Then $P(QQ', RR', \ldots)$ is an thogonal involution pencil. Hence $(QQ', RR', \ldots)$ is an ıvolution on the conic. Hence QQ', RR', ... all pass through ı fixed point, F, say. Now suppose Q to coincide with P, then QQ' coincides with PQ', which is now the normal at P, since PQ is now the tangent at P; hence the normal is a position of QQ' and therefore passes through F.

The point F is called the *Frégier point* of the point P.

For a proof by Reciprocation, see VIII. § 21, Ex. 7.

Ex. 1. *In a parabola, PF is bisected by the axis.*

Take PQ parallel to the axis; then PQ' is bisected by the axis and $Q'F$ is parallel to the axis.

Ex. 2. *In a central conic, the angle PCF is bisected by the axes.*

Take PQ parallel to the minor axis, then F is on CQ.

Ex. 3. *If the chords PQ, PQ' of a conic be drawn equally inclined to the tangent at the fixed point P, then QQ' passes through a fixed point on the tangent at P.*

8. *To construct the double points of an involution range on a line or the double lines of an involution pencil.*

In the case of an involution range on a line, project the range on to any conic through a vertex on the conic; determine the double points of the involution on the conic; then the projections of these double points on the line are the double points of the involution on the line.

In the case of an involution pencil, draw any conic through the vertex, and join the vertex to the double points of the involution which the pencil determines on the conic. These joins are the double lines.

XIX AND XX

1. The conic c_3 touches the conic c_1 at the two points L and M and touches the conic c_2 at the two points N and R. Show that two common chords of c_1 and c_2 meet at the intersection of LM and NR.

2. In a parabola, the locus of the Frégier point is an equal parabola.

3. In a central conic, the locus of the Frégier point is a homothetic conic, the ratio of similitude being

$$a^2 - b^2 : a^2 + b^2$$

where $2a$ and $2b$ are the axes of the given conic.

4. A system of coaxal circles cuts a given circle in pairs of points in involution.

5. Three concurrent chords AA', BB', CC' of a circle are drawn, show that

$$\sin \tfrac{1}{2} AB \sin \tfrac{1}{2} B'C \sin \tfrac{1}{2} C'A' = -\sin \tfrac{1}{2} A'B' \sin \tfrac{1}{2} BC' \sin \tfrac{1}{2} CA$$

where AB denotes the angle subtended by AB at the centre.

6. A, B, C are points on a conic. A', B', C' are points taken on the conic such that

$$(AA', BC) = (BB', CA) = (CC', AB) = -1.$$

Show that (AA', BB', CC'), $(AA', BC', B'C)$, $(BB', AC', A'C)$, and $(CC', AB', A'B)$ are involutions on the conic.

7. A variable circle passes through a fixed point and cuts a given circle at a given angle. Show that it determines on the circle two homographic ranges.

8. A variable circle cuts a given circle and a given line orthogonally. Show that it determines both on the circle and on the line a range in involution.

9. Given two pencils $V(ABC)$ and $V'(A'B'C')$, draw through V and V' a circle meeting the pencils in the points a, b, c, a', b', c' such that aa', bb', cc' is an involution on the circle.

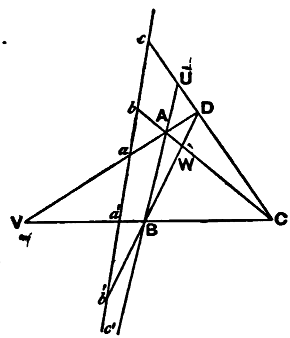
10. Two homographic ranges on equal circles can be placed in involution.

CHAPTER XXI

INVOLUTION OF A QUADRANGLE

1. *The three pairs of points in which any transversal cuts the opposite sides of a quadrangle are in involution.*

Let the transversal meet the sides of the quadrangle $ABCD$ in aa', bb', cc'. Then, considering the pencils subtended at any two vertices A and C of the quadrangle by the range on the line DB joining the other two vertices, we get $A(DWBb') = C(DWBb')$. Hence $(abc'b') = (cba'b')$. Hence $(abc'b') = (a'b'cb)$. Hence $(aa', bb', c'c)$ is an involution, i.e. (aa', bb', cc') is an involution.



For a proof by (1, 1) correspondence, see § 8.

Another proof by Projection is — Project CD to infinity. Then Ab and Ba' are parallel and also Aa and Bb'. Hence

$$c'a' : c'b :: c'B : c'A :: c'b' : c'a;\ \text{hence}\ c'a \,.\, c'a' = c'b \,.\, c'b'.$$

Hence aa', bb' form an involution of which c' is the centre, i.e. c' corresponds to c which is at infinity. Hence in the original figure aa', bb', cc' form an involution.

To determine the mate c' of c in the involution determined by (aa', bb').

Take any point V. On Va take any point A. Let bA cut Va' in C. Let Cc cut VA in D. Let Db' cut VC in B. Then AB cuts aa' in the required point c'.

Ex. 1. *Show that each diagonal of a quadrilateral is divided harmonically.*

Consider CC' as the transversal of the quadrangle $ABA'B'$. Then C, C' are the double points.

Ex. 2. *If through any point parallels be drawn to the three pairs of opposite sides of a quadrangle, a pencil in involution is obtained.*

Consider the involution on the line at infinity.

Ex. 3. *U, V, W are the harmonic points of the quadrangle $ABCD$ of the text. If $U(PQ, BC) = -1 = V(PQ, CA)$, show that $W(PQ, AB) = -1$.*

Consider the involution on PQ.

Ex. 4. *The three meets of any line with the sides of a triangle and the three orthogonal projections of the vertices on this line form an involution.*

Take the fourth vertex at infinity on the projecting lines.

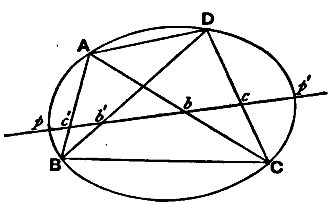
Ex. 5. *If from any point three lines be drawn to the vertices of a triangle, and three other lines parallel to the sides; these six lines form an involution.*

Ex. 6. *A transversal cuts the sides of a triangle ABC in P, Q, R, and PP', QQ', RR' form an involution on the transversal; show that AP', BQ', CR' are concurrent.*

Involution of four-point conics.

2. Desargues's theorem.—*Any transversal cuts a conic and the opposite sides of any quadrangle inscribed in the conic in four pairs of points in involution.*

Let $ABCD$ be the inscribed quadrangle. Let the transversal cut the conic in p, p', AD in a, BC in a', AC in b, BD in b', CD in c, and AB in c'. Then the four points p, p', A, D on the conic subtend homographic pencils at the two points B, C on the conic. Hence $B(pp'AD) = C(pp'AD)$; i.e. $(pp'c'b') = (pp'bc)$; hence $(pp'c'b') = (p'pcb)$. Hence (pp', bb', cc') is an involution. Hence cc' belong to the involution (pp', bb'). Similarly, aa' belong to this involution. Hence (pp', bb', cc', aa') is an involution.



3. *The system of conics which can be drawn through four given points are cut by any transversal in pairs of points in involution.*

For pp' belong to the involution (aa', bb', cc') determined by the opposite sides of the given quadrangle. And similarly for any other conic of the system.

Note that we have above given an independent proof that (aa', bb', cc') is an involution. For through $ABCD$ and any point p on the transversal we can draw a conic.

Note also that we should expect aa', bb', cc' to belong to the involution $(pp', qq', \ldots)$ determined by the conics through the four points. For each pair of opposite sides of the quadrangle is a conic through the four points.

We may prove §§ 1, 2, 3 by (1, 1) correspondence, thus—*Any transversal cuts the three pairs of opposite sides of a quadrangle and the system of conics through the four vertices in pairs of points in involution.*

Take any point P on the transversal l and let the conic through the vertices and P cut l again at P'. Then when P is given, P' is known uniquely. Also when P' is given, P is known uniquely; and by the same construction, for if we draw a conic through P' and the vertices, this cuts l again at P. Also the construction is rational. Hence P and P' generate an involution on l. If P is taken on one of the sides of the quadrangle, this side meets the conic through P and the vertices in three points, and hence the conic breaks up into this side and the opposite side of the quadrangle. Hence l meets this pair of opposite sides (and, similarly, every pair of opposite sides) in a pair of points of the above involution.

For a proof by Projection, see XXIX. 6, Ex. 15.

Ex. 1. *Any transversal cuts a conic in PQ and the successive sides of a four-sided inscribed figure in* 1, 2, 3, 4; *show that*

$$\frac{P1 \cdot P3}{Q1 \cdot Q3} = \frac{P2 \cdot P4}{Q2 \cdot Q4}.$$

Ex. 2. *On every line, there is a pair of points which are conjugate for every one of a system of conics through four given points.*

Viz. the double points of the involution determined by the conics.

Ex. 3. *Through the centres of a system of four-point conics can be drawn pairs of parallel conjugate diameters.*

Take the line in Ex. 2 at infinity.

Ex. 4. *The segment between the points of contact of a common tangent of two conics is divided harmonically by any opposite pair of common chords.*

For each point of contact, being a coincident pair of points in the involution, is a double point.

Ex. 5. *A is the middle point of a chord of a conic; B, C are points on the chord equidistant from A; BDE and CFG are chords of the conic; show that EF and GD cut BC in points equidistant from A.*

Ex. 6. *A transversal parallel to a side of a quadrangle inscribed in a conic cuts the opposite side in O, and the conic and a pair of opposite sides in AA', BB'; show that*

$$OA \cdot OA' = OB \cdot OB'.$$

Ex. 7. *Three sides of a four-sided figure inscribed in a conic pass through three fixed points on a line; show that the fourth side passes through a fourth fixed point on the same line.*

Ex. 8. *Extend the theorem to any $2n$-sided figure.*

Ex. 9. *By taking the two vertices coincident which lie on the $2n$th side, deduce a simple solution of the problem—'Inscribe in a given conic a polygon of $2n-1$ sides, each side to pass through one of a set of $2n-1$ fixed collinear points.'*

Draw tangents from the $2n^{\text{th}}$ fixed point.

Ex. 10. *Show that the problem—'To inscribe in a given conic a polygon of $2n$ sides, each side to pass through one of a set of $2n$ fixed collinear points'—is either indeterminate or impossible.*

For the $2n^{\text{th}}$ fixed point is given by Ex. 8.

4. If A and D become coincident, AD becomes the tangent at A, b and c coincide, and b' and c' coincide. Hence, *if ABC be a triangle inscribed in a conic, and if any transversal cut BC, CA, AB in a', b, b', the tangent at A in a, and the conic in p, p', then pp' is a pair of points in the involution determined by (aa', bb').*

Ex. *A, B are the ends of a diameter of a conic, and C, D*

are fixed points on the conic; find a point P on the conic, such that PC, PD intercept on AB a segment bisected by the centre of the conic.

The tangent at P and CD must meet AB in points equidistant from the centre.

5. If A and B coincide, and also C and D, then a, a', b, b' all lie on AC, at the point E, say; i.e. E is a double point of the involution. Hence, *if any transversal cut a conic in p, p', the tangents at A and C in c, c', and AC in E, then E is a double point of the involution determined by cc', pp'.*

Ex. 1. *The tangents of a conic at P and Q meet in T. A transversal meets the conic in AA', the tangents in BB', and PQ in C; show that $CA \,.\, CB' \,.\, BA' = CA' \,.\, BC \,.\, B'A$.*

Ex. 2. *The tangents at the points PQR on a conic meet in $P'Q'R'$, and the corresponding opposite sides of the triangles PQR, $P'Q'R'$ meet in $P''Q''R''$; show that*

$$(PP'', Q'R'), (QQ'', R'P'), (RR'', P'Q')$$

are harmonic ranges.

Ex. 3. *The tangents of a conic at P and Q meet in T. A transversal parallel to PQ cuts the conic in AA' and the tangents in BB'; show that $AB = A'B'$.*

For one double point is at infinity.

Ex. 4. *Any transversal cuts a hyperbola and its asymptotes in AA', BB'; show that $AB = A'B'$.*

Ex. 5. *The tangents of a conic at P and Q meet in T. A line parallel to QT cuts PT in L, PQ in N, and the conic in M and R. Show that $LN^2 = LM \,.\, LR$.*

6. If A, B and C coincide, then a', c' and b coincide, and a, b' and c coincide. Hence, *if a system of conics be drawn having three-point contact at A, and passing through D, then any transversal cuts the conics in pairs of points in involution, one pair being the points on AD and on the tangent at A.*

Ex. *The common tangent of a conic and its circle of curvature at P is divided harmonically by the tangent at P and the common chord.*

7. If A, B, C and D coincide, then a, a', b, b', c, c' all coincide in the point E, where the tangent at A cuts the

transversal. Hence, *a system of conics having four-point contact at a point is cut by any transversal in pairs of points in involution, of which one double point is on the tangent at the point.*

Ex. 1. *The tangent at the point R to the circle of curvature at the vertex of a conic cuts the conic in P, Q, and the tangent at the vertex in T. Show that* $(TR, PQ) = -1$.

Ex. 2. *If two conics have four-point contact at a point, the polars of any point on the tangent at this point coincide.*

8. *If a transversal cut two pairs of opposite sides of the quadrangle $ABCD$ in aa', bb', and any two corresponding points p, p' be taken in the involution (aa', bb'); then the six points A, B, C, D, p, p' lie on a conic.*

For draw a conic through $ABCDp$; then the conic passes also through p' by 'reductio ad absurdum.'

Ex. 1. *A line cuts two conics in AB, $A'B'$, and E, F are the double points of the involution AA', BB'; show that a conic through the meets of the given conics can be drawn through E, F.*

Ex. 2. *AB, BC, CD, DA touch a conic. Through U (the meet of AC, BD) is drawn any chord PQ of the conic; show that the six points A, B, C, D, P, Q lie on a conic.*

Ex. 3. *A conic passes through three out of four vertices of a quadrangle, and a line meets the six sides and the conic in pairs of points of an involution. Show that the conic also passes through the fourth vertex.*

XXI

1. Six points A, B, C, A', B', C' are taken, and through any point O are drawn OP, OP', OQ, OQ', OR, OR' parallel to AA', BC, BB', CA, CC', AB. If the angles POP', QOQ', ROR' have the same bisectors, then AA', BB', CC' are concurrent.

2. Any line cuts a conic at P and Q and the six successive sides of a hexagon inscribed in the conic at the points 1, 2, 3, 4, 5, 6. Show that

$$\frac{P1 \,.\, P3 \,.\, P5}{Q1 \,.\, Q3 \,.\, Q5} = \frac{P2 \,.\, P4 \,.\, P6}{Q2 \,.\, Q4 \,.\, Q6}.$$

3. If two conics c_1 and c_2 are so situated that triangles

can be inscribed in c_2 which are self-conjugate for c_1, then the pole for c_1 of any triangle inscribed in c_2 lies on c_2.

4. Through the fixed point B on a hyperbola are drawn the lines BP, BQ parallel to the asymptotes. Through the fixed point O on the hyperbola is drawn the variable chord $OQPR$ cutting the conic again at R. Show that the ratio $PR:QR$ is constant.

5. Two parabolas with parallel axes touch at P. A transversal is drawn cutting the tangent at P at the point O, the diameter through P at E, and the curves at Q, Q' and R, R'. Show that $OE^2 = OQ \,.\, OQ' = OR \,.\, OR'$.

6. If two conics touch and if the two polars of every point on the common tangent coincide, show that the conics have four-point contact. Also deduce the theorem that two equal parabolas which have the same axis have four-point contact at infinity.

7. $ABCD$, $A'B'C'D'$ are two quadrangles inscribed in the same conic. $A'B'$, $C'D'$ meet AD, BC at E, F, G, H. $A'D'$, $B'C'$ meet AB, CD at E', F', G', H'. Show that the eight points E, F, G, H, E', F', G', H' lie on the same conic.

8. Four points A, B, C, D are taken on a circle. AB cuts another circle at A', B' and CD cuts this circle at C', D'. BD cuts $A'D'$, $B'C'$ at E, F; and AC cuts $A'D'$, $B'C'$ at H, G. Show that E, F, G, H lie on a coaxal circle.

9. O is a fixed point on a given conic of a system of conics through four given points. The tangent at O cuts any one of the system at the points P and Q. Show that $OP^{-1} + OQ^{-1}$ is constant.

10. Two conics $PQRS$ and $P'Q'R'S'$ are drawn to touch four given lines PP', QQ', RR', SS'. Prove that if a pair of common chords is orthogonal, then one pair of opposite sides of the quadrangle $PQRS$ are inclined at the same angle as the corresponding pair of the quadrangle $P'Q'R'S'$.

CHAPTER XXII

POLE-LOCUS AND CENTRE-LOCUS

1. *The polars of a given point for a system of four-point conics are concurrent.*

Let X be the given point. Let the polars of X for two conics α, β of the system meet in X'. Consider the involution $(pp', qq', rr', \ldots)$ determined by the conics α, β, γ, ... of the system on the line XX'. Since

$$(XX', pp') \text{ and } (XX', qq')$$

are harmonic, XX' are the double points of the involution. Hence (XX', rr'), &c., are harmonic. Hence XX' are conjugate points for every conic of the system. Hence the polars of X for the system are concurrent in X'.

X, X' are called *conjugate points for the system of four-point conics.*

Ex. 1. *Of a system of four-point conics, the diameters bisecting chords in a fixed direction are concurrent.*

Here X is at infinity.

Ex. 2. *The polars of a given point for the three pairs of opposite sides of a quadrangle are concurrent.*

For each pair is a conic.

2. *Given a system of four-point conics and a line l, the locus of the poles of l for conics of the system is a conic, which coincides with the locus of points which are conjugate to points on l for conics of the system.*

Let the poles of l for conics α, β, γ, ... of the system be L, M, N, ... ; and let X', Y', ... be the conjugate points of the points X, Y, ... on l for the system. Then the polars of X, Y, ... for α are LX', LY', ... Hence

$$(XY \ldots) \doteq L(X'Y' \ldots).$$

So $(XY\ldots) = M(X'Y'\ldots)$. Hence $L(X'Y'\ldots) = M(X'Y'\ldots)$. Hence $LMX'Y'\ldots$ lie on a conic. Hence all the points $X'Y'\ldots$ lie on a conic which passes through L and M. Similarly the locus passes through $N\ldots$. Hence all the points $LMN\ldots$ and all the points $X'Y'\ldots$ lie on a single conic, called *the pole-locus of the line l for the system of four-point conics*.

The pole-locus is also called *the eleven-point conic* because it passes through eleven points which can be constructed at once from the given line and the given quadrangle.

Three of these points are the harmonic points of the quadrangle. For U is conjugate to the point in which VW cuts l; and so on.

Six more of these points are the fourth harmonics of a for AD, b for AC, c for DC, a' for BC, b' for BD, c' for BA, taking the transversal of the figure of XXI. 1 as l. For the polar of a for every conic of the system passes through the fourth harmonic of a for AD, since A and D are on the conic.

The last two points are the double points of the involution determined by the conics and the quadrangle on l. For these are conjugate for each conic of the system, being harmonic with the points in which l cuts the conic.

Notice that for a given quadrangle the eleven-point conics form a system of three-point conics; for they all pass through U, V, W.

Ex. 1. *If the quadrangle be a square, the pole-locus is a rectangular hyperbola.*

Ex. 2. *If l pass through one of the harmonic points of the given quadrangle, the pole-locus breaks up into a pair of lines.*

Let l pass through W. Then UV contains four of the eleven points, viz. U, V and the fourth harmonics of W for AC and BD. Hence the locus cannot be a curved conic; hence it is two lines.

Ex. 3. *The polars of any two points for conics of a four-point system form two homographic pencils.*

For $X'(LMN\ldots) = Y'(LMN\ldots)$.

Ex. 4. *The pencil of tangents at one of the four common points of a system of four-point conics is homographic with that at any other of the four points.*

A particular case of Ex. 3.

3. Taking l at infinity we deduce the following theorem—*The locus of the centres of a system of conics circumscribing a given quadrangle is a conic which passes through the harmonic points of the quadrangle, through the middle points of the six sides of the quadrangle, and through the common conjugate points for the system on the line at infinity.*

Notice that the centre-locus also passes through the conjugate point for the system of every point at infinity.

If the quadrangle is re-entrant, it is easy to see that the sides of the quadrangle cut the line at infinity in an overlapping involution. Hence the common conjugate points at infinity, i.e. the points at infinity on the centre-locus, are imaginary, and the centre-locus is an ellipse. So if the quadrangle is not re-entrant, the centre-locus is a hyperbola.

Ex. 1. *Five points $ABCDE$ are taken. Show that the five conics which bisect the sides of the five quadrangles $BCDE$, $ACDE$, $ABDE$, $ABCE$ and $ABCD$ meet in a point.*

Viz. the centre of the conic $ABCDE$.

Ex. 2. *If a pair of opposite sides of the quadrangle be parallel, the centre-locus is a pair of lines.*

Ex. 3. *If two pairs of sides, not opposite, be parallel, the centre-locus is a line (and the line at infinity).*

Ex. 4. *From a variable line are cut off by two given conics lengths which are bisected at the same point P. Show that the locus of P is the centre-locus belonging to the meets of the conics.*

For the centre-locus is the locus of points conjugate to those on the line at infinity.

Ex. 5. *The polars of any point on the centre-locus for conics of the system are parallel.*

Ex. 6. *The asymptotes of any conic of the system are parallel to a pair of conjugate diameters of the centre-locus.*

Let O be the centre of that conic of the system which meets the line at infinity in pp'. Now the centre-locus

meets the line at infinity in the double points e, f of the involution $(pp', \ldots)$. Hence $(pp', ef) = -1$. Hence

$$Z(pp', ef) = -1$$

where Z is the centre of the centre-locus. But Ze, Zf are the asymptotes of the centre-locus. Hence Zp, Zp' are conjugate diameters of the centre-locus. And Zp, Zp' are parallel to Op, Op', which are the asymptotes of the conic whose centre is O.

Ex. 7. *If one of the four-point conics be a circle, the centre-locus is a rectangular hyperbola.*

For the common conjugate points at infinity, being conjugate for a circle, subtend a right angle at any finite point, i.e. the asymptotes of the centre-locus are perpendicular.

Ex. 8. *The locus of the centres of rectangular hyperbolas circumscribing a given triangle is the nine-point circle.*

Ex. 9. *The centre of the centre-locus of $ABCD$ is the centroid of A, B, C, D.*

For if P, P', Q, Q', R, R' bisect AB, CD, AD, BC, AC, BD, then PQ and $P'Q'$ are parallel to BD, and PQ' and $P'Q$ are parallel to AC. Hence $PQP'Q'$ is a parallelogram. Hence PP', QQ' intersect at the centre of the centre-locus, i.e. the centre bisects PP' (and similarly QQ', RR').

Ex. 10. *The six fourth harmonics of the ends of the six sides of a quadrangle for the intersections with any transversal are joined in opposite pairs. Show that the connectors are concurrent.*

XXII

1. If an intersection of opposite sides of a given quadrangle is joined to the middle point of the segment cut off from a given transversal by these sides, the three lines so formed are concurrent.

2. The polars of a given point for a system of conics touching two given lines at given points meet at a point on the chord of contact.

3. If l passes through A, then the pole-locus of l for the quadrangle $ABCD$ touches l at A; and if l passes through A and C, the pole-locus is l and another line.

4. The locus of the centres of all conics through the vertices of a triangle and its centroid is the maximum inscribed ellipse.

5. If a pair of sides of the quadrangle which are not opposite are parallel, the centre-locus is a parabola.

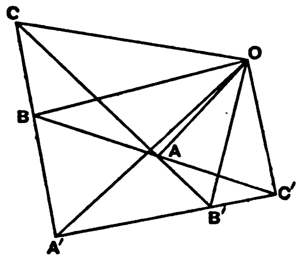
6. The nine-point circles of the four triangles formed by four points meet in a point.

7. A system of conics have three-point contact at A and pass through B. Show that the centres of the conics lie on a conic whose centre O is such that $3 . AO = OB$.

CHAPTER XXIII

INVOLUTION OF A QUADRILATERAL

1. *The three pairs of lines which join any point to the opposite vertices of a quadrilateral are in involution.*



This is proved by reciprocating—'The three pairs of points in which any transversal cuts the opposite sides of a quadrangle are in involution'; which is proved in XXI. 1.

To determine the mate OC' of OC in the involution determined by $O(AA', BB')$.

Draw any line cutting OC, OB, OA' at C, B, A'. Through C draw any line cutting OA, OB' at A, B'. Let AB and $A'B'$ meet at C'. Then OC' is the required ray. For AA', BB', CC' are the opposite vertices of a complete quadrilateral.

Ex. 1. *If any point O be joined to the vertices ABC of a triangle, and if OA', OB', OC' be drawn parallel to BC, CA, AB, then $O(AA', BB', CC')$ is an involution.*

Ex. 2. *If any point O be joined to the vertices ABC of a triangle, and $A'B'C'$ be points on the sides of the triangle, such that $O(AA', BB', CC')$ is an involution; then $A'B'C'$ are collinear.*

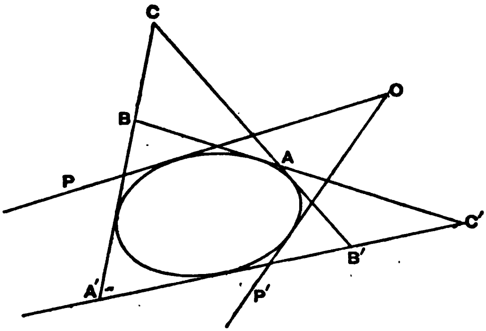
Ex. 3. *The orthogonal projections of the vertices of a quadrilateral on any line are in involution.*

Ex. 4. *An infinite number of pairs of lines can be found which divide all three diagonals of a quadrilateral harmonically.*

The pair of lines through any point O are the double lines of the involution $O(AA', BB', CC')$.

Involution of four-tangent Conics.

2. *The pair of tangents from any point to a conic and the pairs of lines joining this point to the opposite vertices of any quadrilateral circumscribing the conic are four pairs of lines in iuvolution.*



This is proved by reciprocating—'Any transversal cuts a conic and the opposite sides of any quadrangle inscribed in the conic in four pairs of points in involution'; which is proved in XXI. 2.

3. *The system of conics which can be drawn to touch four given lines is such that the pairs of tangents from any point to conics of the system form an involution.*

For the tangents OP, OP' to any conic of the system belong to the involution $O(AA, BB', CC')$, determined by the opposite vertices of the given quadrilateral of tangents.

Note that we have above given an independent proof that $O(AA', BB', CC')$ is an involution. For touching the four given lines and any other line OP we can draw a conic.

Note also that we should expect OA, OA'; OB, OB'; OC, OC' to belong to the involution $O(PP', QQ', \ldots)$ of tangents. For each pair of opposite vertices may be considered to be a conic which touches the four lines; and OA, OA' are the tangents from O to the conic (A, A').

Or we may prove §§ 1, 2, 3 by (1, 1) correspondence, thus—

Take any line OP through the fixed point O. Then we

can draw a conic to touch OP and the four given lines. Let OP' be the other tangent to this conic from O. Then when OP is given, OP' is known uniquely; and when OP' is given, OP is known uniquely, and by the same construction. This is clearly rational; hence OP and OP' generate an involution at O. Also the pairs AA', BB', CC' of intersections of opposite sides of the quadrilateral are degenerate conics of the system, and hence OA, OA' and OB, OB' and OC, OC' belong to the involution of pairs of tangents from O.

4. If two sides BA and AB' coincide, we get the theorem—*If a triangle $BA'B'$ be circumscribed to a conic, and if A be the point of contact of BB'; then the tangents from O are a pair in the involution $O(AA', BB')$.*

If the sides CB and $C'B$ coincide and also the sides CB' and $C'B'$, we get the theorem—*If a conic touch the lines CB, CB' at B and B', then the tangents from O are a pair in the involution $O(CC, BB')$ of which OC is a double line.*

If the sides BA, AB' and $B'A'$ coincide, we get the theorem—*If a system of conics have three-point contact at B', and if the common tangent at B' cut, at B, a second common tangent, then the tangents from O form an involution of which OB, OB' are a pair.*

For three-point contact and three-tangent contact are equivalent.

If all four sides coincide, we get the theorem—*The tangents from O to a system of conics having four-point contact at a point B' form an involution of which OB' is a double line.*

Ex. 1. *If the line joining the centres of similitude SS' of two circles cut the circles in AA', BB'; then AA', BB', SS' are in involution.*

Take O at infinity in a direction perpendicular to SS'.

Ex. 2. *Through every point can be drawn a pair of lines which are conjugate for every conic of a four-tangent system.*

Viz. the double lines of the involution of tangents.

Ex. 3. *The tangents at one of the intersections of two conics*

inscribed in the same quadrilateral are harmonic with the lines joining the point to any two opposite vertices of the quadrilateral.

For the tangents are the double lines.

Ex. 4. *ABC is a triangle and O a given point. Through O, and parallel to the sides BC, CA, AB, are drawn the lines OX, OY, OZ; show that the double lines of the involution $O(XA, YB, ZC)$ are the tangents at O to the two parabolas which can be inscribed in ABC so as to pass through O.*

Ex. 5. *P, Q, R are the points of contact of the lines BC, CA, AB with a conic, and OT, OT' are the tangents from any point O; show that $O(BC, PA, TT')$ and $O(RQ, AA, TT')$ are involutions.*

Consider $ABPCA$ and $ARAQA$.

Ex. 6. *Three vertices of a four-point figure circumscribed to a conic lie on three fixed lines through a point; show that the fourth vertex lies on a fourth fixed line through the same point.*

5. *The three circles on the diagonals of any quadrilateral as diameters are coaxal.*

The three middle points of the diagonals of a quadrilateral lie on a line (called *the diameter of the quadrilateral*).

The directors of a system of conics touching the sides of a quadrilateral are coaxal, and three circles of the coaxal system are the three circles on the diagonals as diameters.

The centres of a system of conics touching the sides of a quadrilateral lie on a line which also contains the middle points of the diagonals of the quadrilateral.

Let AA', BB', CC' be the opposite vertices of the quadrilateral. Let the circles on AA' and BB' as diameters meet in O and O'. Then in the involution pencil $O(AA', BB', CC')$, since AOA' and BOB' are right angles, COC' is a right angle. Hence the circle on CC' as diameter passes through O; and similarly through O'. Hence the circles on AA', BB', CC' as diameters are coaxal. Hence their centres, viz. the middle points of AA', BB', CC', are collinear.

Again, the tangents OP, OP' from O to any conic touching the sides of the quadrilateral belong to the involution $O(AA', BB', CC')$. Hence POP' is a right angle. Hence

the director of this conic passes through O; and similarly through O'. Hence this director, and similarly all the directors, belong to the above coaxal system. But the centre of a conic is the same as the centre of its director. Hence the centres of the conics lie on a line, viz. the line of centres of the coaxal system of circles.

The locus of centres is the diameter of the quadrilateral; for three circles of the system are the circles on AA', BB', CC' as diameters.

The radical axis of the coaxal system of directors is the directrix of the parabola of the system of conics.

For when the director becomes a straight line, it becomes the radical axis of the coaxal system to which the director belongs, and it also becomes the directrix of the parabola touching the four lines.

The limiting points of the coaxal system of directors are the centres of the rectangular hyperbolas of the system of conics.

For when the director becomes a point, it becomes one of the limiting points of the coaxal system to which the director belongs, and it also becomes a centre of a rectangular hyperbola touching the four lines; for in a rectangular hyperbola only is the director a point, viz. the centre (since we must have $a^2 + b^2 = 0$).

Note that the director of a conic which consists of two points is the circle on the segment joining the points as diameter, and the centre of the conic is the point half-way between the points; and if the two points coincide, the director becomes the point itself.

Ex. 1. *The directors of all conics touching two given lines OP, OQ at P, Q are coaxal, the axis being the radical axis of the point O and the circle on PQ as diameter.*

Ex. 2. *The polar circle of a triangle circumscribing a conic is orthogonal to the director circle.*

Call the triangle ABB', and let BB' touch at A'. Then $ABA'B'A$ is a circumscribed quadrilateral. Hence the director circle and the circles on AA' and BB' as diameters are coaxal. Hence a circle orthogonal to the last two circles is also orthogonal to the director circle. But A, A' and

also B, B' are conjugate points for the polar circle; hence the polar circle is orthogonal to the circles on AA' and BB' as diameters.

Ex. 3. *The locus of the centre of a rectangular hyperbola which touches a given triangle is the polar circle of the triangle.*

For the polar circle cuts orthogonally the director circle which is the centre in the case of a rectangular hyperbola.

Ex. 4. *The diameters of the five quadrilaterals which can be formed by five given lines are concurrent. Prove this, and deduce a construction for the centre of a conic, given five tangents.*

Ex. 5. *The axis of the parabola inscribed in a quadrilateral is parallel to the diameter of the quadrilateral.*

Ex. 6. *The diameter of a quadrilateral circumscribing a conic touches the centre-locus of the quadrangle formed by the points of contact.*

Otherwise the conic would have two centres.

Ex. 7. Steiner's theorem. *The orthocentre of a triangle circumscribing a parabola is on the directrix.*

Let the sides BC, CA, AB of the triangle cut the line at infinity at A', B', C'. Then, if O is the orthocentre, OA is perpendicular to BC and therefore to OA' (which is parallel to BC); so for OB, OB' and OC, OC'. Hence the involution $O(AA', BB', CC')$ at O is orthogonal. Hence the tangents from O are orthogonal. Hence O is on the directrix.

Ex. 8. Gaskin's theorem. *The circle circumscribing a triangle which is self-conjugate for a conic is orthogonal to the director circle of the conic.*

Take any tangent to the conic. Then from this tangent and the given self-conjugate triangle UVW, we can construct three other tangents such that UVW is the harmonic triangle of the quadrilateral so formed. Let AA', BB', CC' be the opposite vertices of this quadrilateral.

Then the circle about UVW is clearly orthogonal to the circles on AA', BB', CC' as diameters, for it cuts these diameters in the inverse points, VW, WU, and UV. Hence the circle about UVW, being orthogonal to three circles of a coaxal system, is orthogonal to the director, which belongs to the coaxal system.

Ex. 9. *The centre of a circle circumscribing a triangle self-conjugate for a parabola is on the directrix.*

Ex. 10. *The circle circumscribing a triangle self-conjugate for a rectangular hyperbola passes through the centre.*

Ex. 11. *Show that two, and only two, rectangular hyperbolas can be drawn to touch four given lines.*

Let the lines be a, b, c, d. Let the circle about the harmonic triangle of the quadrilateral a, b, c, d meet the diameter of the quadrilateral in L and L'. Then L and L' are the limiting points of the directors, being the intersections of a circle orthogonal to the coaxal system of directors with the line of centres.

First take L, and let a' be the reflexion of a in L. Construct the conic touching a, b, c, d, a'. Then the centre of this conic, being the meet of the diameter of the quadrilateral and the line half-way between a and a', is L. Hence L is the centre of the director. But the coaxal with centre at L has zero radius. Hence the conic is a r. h.

So L' gives another r. h. And there are only two; for the centre must be at L or at L' by Ex. 10.

Ex. 12. *Any transversal cuts the diagonals AA', BB', CC' of a quadrilateral circumscribed to a conic in the points P, Q, R, and points P', Q', R' are taken such that (AA', PP'), (BB', QQ'), (CC', RR') are harmonic; show that $P'Q'R'$ and the pole of the transversal for the conic are collinear.*

Project PQR to infinity.

6. *The locus of the poles of a given line for a system of four-tangent conics is a line.*

This is proved by reciprocating—'The polars of a given point for a system of four-point conics are concurrent' which is proved in XXII. 1.

Taking the given line at infinity, we again see that—

The locus of the centres of a system of four-tangent conics is a line.

By reciprocating the properties of the pole-locus (or directly) we can investigate the properties of *the polar-envelope of a point for a system of four-tangent conics.*

XXIII

1. The six radical axes of four circles through the same point form an involution.

2. AA', BB', CC' are the opposite vertices of a quadrilateral. Show that

$$AB \,.\, AB' \div AC \,.\, AC' = A'B \,.\, A'B' \div A'C \,.\, A'C'.$$

Also if P, Q, R bisect AA', BB', CC', then

$$AB \,.\, AB' \div AC \,.\, AC' = PQ \div PR.$$

3. Show that the problem—'To circumscribe to a given conic a polygon of $2n$ vertices, each vertex to lie on one of a set of $2n$ fixed concurrent lines'—is either indeterminate or impossible.

4. Circumscribe to a given conic a polygon of $2n-1$ vertices, each vertex to lie on one of a set of $2n-1$ fixed concurrent lines.

5. If the nine-point circle of a triangle circumscribing a rectangular hyperbola passes through the centre of the rectangular hyperbola, show that the centre also lies on the circumcircle, and that the centre of the circumcircle lies on the rectangular hyperbola.

6. The directrices of all parabolas touching the sides of a given triangle are concurrent.

7. Given five points on a conic, five self-conjugate triangles can be found, viz. the harmonic triangles of the inscribed quadrangles obtained by omitting one point. Show that the ten radical axes of the circles circumscribing these triangles pass through the centre of the conic.

8. The side BC of the triangle ABC circumscribed to a circle touches at P. If L bisects AP and M bisects BC, show that the centre of the circle lies on LM.

CHAPTER XXIV

CONSTRUCTIONS OF THE FIRST DEGREE

1. *Examples of constructions in which the ruler only is to be used.*

Ex. 1. *Given the segment AC bisected in B; prove the following construction for a parallel to AC through P—Through B draw any line, cutting PA in E and PC in D; then if CE, DA meet in Q, PQ is the required line.*

Let PQ cut AC at R. Then (BR, AC) is harmonic. But B bisects AC. Hence R is at infinity.

Ex. 2. *Given two parallel segments AB and CD, prove the following construction for bisecting each—Let CB, AD meet in W, and AC, BD in V, then VW bisects both segments.*

For U is at infinity.

Ex. 3. *Given a pair of parallel lines, draw through a given point a parallel to both.*

Use Ex. 2 and then Ex. 1.

Ex. 4. *Given a parallelogram, bisect a given segment.*

Let AB be the segment. Through A and B draw parallels to the sides of the parallelogram, meeting again in C and D. Then CD bisects AB.

Ex. 5. *Given two lines AB and CD which meet in an inaccessible point U, construct any number of points on the line joining U to a given point O.*

Through O draw LOM' and MOL' meeting AB in LM and CD in $L'M'$. Let LL', MM' meet in W. Then $U(AC, OW)$ is harmonic; hence the required line is the polar of W for AB and CD. To construct any other point on the line, draw any two lines WNN' and WRR' meeting AB in N, R, and CD in N', R'. Then a point on the required line is the meet of NR' and $N'R$.

Ex. 6. *Construct lines which shall pass through the meet of a given line with the line joining two given points, when this last line cannot be drawn.*

Use the construction reciprocal to that of Ex. 5.

Ex. 7. *Given a segment AC bisected at B, join any point P_1 to ABC, on P_1B take any point Q, join CQ cutting AP_1 in L_1, join AQ cutting CP_1 in L_3, join L_1L_3 cutting BP_1 in L_2; then $L_1L_2 = L_2L_3$, and L_1L_3 is parallel to AC. Again, let AL_2, BL_3 cut in P_2, and let CP_2 cut L_1L_3 in L_4; then $L_2L_3 = L_3L_4$. Again, let AL_3, BL_4 cut in P_3, and let CP_3 cut L_1L_4 in L_5; then $L_3L_4 = L_4L_5$. And so on.*

The first part comes from the quadrilateral $P_1L_1QL_3P_1$. The rest follows by Elementary Geometry.

This enables us *to divide a bisected segment into any number of equal parts.* To divide AC into n equal parts, construct the points $L_1L_2 \dots L_{n+1}$. Let AL_1 and CL_{n+1} meet in V. With V as vertex project $L_1L_2 \dots L_{n+1}$ on to AC.

2. *To construct a five-point conic.*

Let A, B, C, D, E be the five given points on the conic. We shall construct the conic by finding the point in which any line AG through A meets the conic again. (See figure of XV. 1.) Let AG and CD meet in L, and AB and DE in M. Let LM cut BC in N. Then, by Pascal's theorem, NE cuts AG in the required point F on AG. And since AG is any line through A, we shall thus construct every point on the conic.

If any two of the points are coincident, the necessary modification of this construction is obvious, remembering that to be given two coincident points is to be given a point and the tangent at the point, and that the two coincident points lie on the tangent.

The case of three or four points being coincident is discussed in XXV. 17.

To construct the polar of a given point for a five-point conic.

Let P be the given point. Construct the points A' and B' in which PA and PB cut the conic again. Then the polar passes through $(AB'; A'B)$ and $(AB; A'B')$.

To construct the pole of a given line for a five-point conic.

Take two points on the line; then the polars of these points will intersect at the required pole.

Ex. *Given a pair of conjugate diameters in magnitude and position, construct the conic by the ruler only.*

By § 1, Ex. 1, we can construct the tangents at P and D, meeting at E. Then if D, D' are real we have five points $PPP'DD'$; and if D, D' are imaginary we have PPP' and two points at infinity on the asymptote CE.

***3.** As an example of coincident points, let us *construct a conic to touch two given lines at given points, and to pass through a given point.*

Suppose the conic is to touch OP and OQ at P and Q, and to pass through A. Here B and C coincide with P, and the line BC coincides with OP. So D and E coincide with Q, and DE coincides with OQ. Hence the construction is—To find where any line AG through A cuts the conic again, let AG and PQ meet in L, and AP and OQ in M; let LM cut OP in N; then NQ cuts AG in the required point F.

Ex. *Given four points and the tangent at one of them, construct the conic.*

***4.** As an example of cases in which some of the given points are at infinity, let us *construct a conic, given one asymptote, the dirction of the other asymptote, and two other points.*

Let l be the given asymptote, and m any line in the direction of the other asymptote, and A and B the two given points. We may take C and D to be the points at infinity on l, and E to be the point at infinity on m. Then M is the point at infinity on AB.

Hence the construction is—To find where any line AG through A cuts the conic again, let AG and l meet in L, and let a parallel through L to AB cut a parallel through B to l in N. Then a parallel through N to m cuts AG in the required point F.

Ex. 1. *Given three points on a conic and a tangent at one of them, and the direction of one asymptote; construct the conic.*

Ex. 2. *Given three points and the directions of both asymptotes, construct the conic.*

Ex. 3. *Given four points on a conic and the direction of one asymptote; construct the meet of the conic with a given line drawn parallel to the asymptote.*

***5.** As an example of drawing a parabola to satisfy given conditions, let us *construct a parabola, given three points and the direction of the axis.*

Let ABC be the given points, and l any line in the direction of the axis. We may consider D and E to coincide at the point at infinity upon l, so that the line DE is the line at infinity. Then M is the point at infinity on AB.

Hence the construction is—To find where any line AG through A cuts the conic again, let AG cut a parallel through C to l in L; let a parallel through L to AB cut BC in N; then a parallel through N to l cuts AG in the required point F.

Ex. *Construct a parabola, given two points and the tangent at one of them, and the direction of the axis.*

6. *Given five points on a conic, to construct the tangent at one of them.*

Let A, B, C, D, E be the five given points, and suppose F to coincide with A; then AF is the tangent at A. Hence the construction—Let AB and DE meet in M, and BC and AE in N, and let MN cut CD in L; then LA is the tangent at A.

Ex. 1. *Given three points on a conic, and the tangents at two of them; construct the tangent at the third.*

Ex. 2. *Given three points on a parabola, and the direction of the axis; construct the tangent at one of the given points.*

Ex. 3. *Given four points on a conic, and the direction of one asymptote; construct that asymptote.*

7. *Given five tangents of a conic, to construct the points of contact.*

Let AB, BC, CE, EF, FA be the five given tangents. Then in the figure of XV. 4, if D is the point of contact of CE, we may consider CD, DE to be consecutive tangents of the conic. Hence the construction—Let BE and CF meet in O; then AO cuts CE in its point of contact. So the other points of contact can be constructed.

Hence given five tangents, we can at once construct five

points; so that every construction which requires five points to be given, is available if we are given five tangents.

Ex. 1. *Given four tangents and the point of contact of one of them, construct the points of contact of the others.*

Ex. 2. *Given four tangents of a parabola, construct the points of contact, and the direction of the axis.*

Ex. 3. *Construct the polar of a given point and the pole of a given line for a five-tangent conic.*

***8.** *Given five tangents of a conic, to construct the conic by tangents.*

Let GB, BC, CD, DE, EH be the given tangents. Now every tangent cuts GB. Hence if we construct every other tangent from points on GB, we shall have constructed every tangent of the conic. On GB take any point A. Let AD and BE meet in O. Let CO meet EH in F. Then, by Brianchon's theorem, FA touches the conic, i.e. AF is the other tangent from any point A on GB.

Ex. 1. *Given five tangents of a conic, construct the tangent parallel to one of them.*

Ex. 2. *Given four tangents of a parabola, construct the tangent in a given direction.*

9. *Given three points on a conic and a pole and polar, to construct the conic.*

Let A, B, C be the three given points, and O the pole. Let OA cut the polar in P, and take A' such that (OP, AA') is harmonic. Similarly construct Q and B'. Through $ABCA'B'$ construct a conic. This will be the required conic; for since (OP, AA') and (OQ, BB') are harmonic, we see that PQ is the polar of O.

A reciprocal construction enables us to solve the problem—*Given three tangents of a conic, and a pole and polar, to construct the conic.*

A simple case of each problem is—*Given three points* (*or three tangents*) *and the centre, to construct the conic.*

We obtain two more points (or tangents) by reflexion in the centre.

CHAPTER XXV

CONSTRUCTIONS OF THE SECOND DEGREE

1. *Construct the points in which a given line cuts a conic given by five points.*

Let A, B, C, D, E be the five given points. Let the given line cut DA, DB, DC in a, b, c, and cut EA, EB, EC in a', b', c', and cut the conic in x, y. Then

$$(xyabc) = D(xyABC) = E(xyABC) = (xya'b'c').$$

Hence x, y are the common points of the two homographic ranges determined by (abc) and $(a'b'c')$. Hence the two required points x, y can be constructed by XVI. 6.

If the given line is the line at infinity, we have to *construct the directions of the asymptotes in the case of a hyperbola, or the direction of the axis in the case of a parabola.*

To do this, take any point V; and draw Va, Vb, Vc, Va', Vb', Vc' parallel to DA, DB, DC, EA, EB, EC. Then the required directions are those of the common lines Vx, Vy of the homographic pencils determined by $V(abc)$ and $V(a'b'c')$. For taking a, b, c, a', b', c', x, y on the line at infinity, we have $(xyabc) = V(xyabc) = V(xya'b'c') = (xya'b'c')$.

Ex. *Through a given point O draw a line meeting four given lines a, a', b, b' at points A, A', B, B', such that*

$$OA \,.\, OA' = OB \,.\, OB'.$$

Through O draw a parallel to either asymptote of the conic through the five points ab, $a'b$, ab', $a'b'$ and O.

2. *Given five tangents to a conic, to construct the tangents from any point to the conic.*

Let three of the given tangents cut the other two in ABC and $A'B'C'$. If a tangent from the given point P cut these tangents in X and X', then $(ABCX) = (A'B'C'X')$; hence $P(ABCX) = P(A'B'C'X')$. But PX and PX' coincide;

hence one of the tangents from P is one of the common lines of the pencils $P(ABC)$ and $P(A'B'C')$. Hence the required tangents are the common lines of the homographic pencils determined by $P(ABC)$ and $P(A'B'C')$.

3. *Given five tangents to a conic, to construct the points in which any line cuts the conic.*

Construct first by XXIV. 7 the points of contact, and then proceed by § 1.

Given five points on a conic, to construct the tangents from any point.

Construct first by XXIV. 6 the tangents at the points, and then proceed by § 2.

4. If instead of five points, we are given four points and the tangent at one, or three points and the tangents at two of them; or if, instead of five tangents, we are given four tangents and the point of contact of one, or three tangents and the points of contact of two of them, the necessary modifications of the above constructions are obvious.

Ex. 1. *Construct a line to be cut by four given lines in a given cross ratio and to pass through a given point.*

Let three (b, c, d) of the lines cut the fourth a in B, C, D. Take A such that $(ABCD)$ is equal to the given cross ratio. Draw a conic to touch the three given lines and also to touch the fourth at A. Through the given point draw a tangent to this conic. This is the required line. There are therefore two solutions.

Ex. 2. *Give the reciprocal construction.*

Ex. 3. *Through a given point draw a line to be cut by three given lines in A, B, C, so that $AB:BC$ is a given ratio.*

5. *Given five points on a conic, to construct the centre, the axes, and the asymptotes.*

Let A, B, C, D, E be the five given points. Through A draw AG parallel to BC, and construct, by Pascal's theorem, the point A' in which AG cuts the conic again. Let AC and BA' cut in H, and AB and $A'C$ cut in K. Then HK cuts AA' at the fourth harmonic for AA' of the intersection

of AA' and BC, i.e. bisects AA'; so it bisects BC. Hence HK is a diameter. Similarly construct another diameter. Then these diameters meet in the centre, O.

To construct the axes and asymptotes, we must first construct the involution of conjugate diameters. To do this—Through the centre O draw Oa parallel to BC, and let Oa' be the diameter bisecting AA' and BC. Then Oa, Oa' are a pair of conjugate diameters. In the same way determine another pair Ob, Ob'. Then the rectangular pair of the involution determined by $O(aa', bb')$ are the axes; and the double lines of the same involution are the asymptotes.

If the diameters are parallel, the conic is a parabola; and the direction of the diameters is the direction of the axis of the parabola.

To construct the foci of a central five-point conic.

Construct the axes as above, and construct the points A, A' and B, B' in which they cut the conic. If A, A' and B, B' are real, on AA' and BB' as diameters construct circles. One of these circles, say that on AA', is outside the conic. Construct the tangent at one of the given points, cutting this circle at Y, Y'. Then the perpendiculars to YY' at Y, Y' will cut AA' in the foci. If B, B' are imaginary, the conic is a hyperbola; and we proceed, as above, with the circle on AA' as diameter.

To construct the axis and focus of a five-point parabola.

Construct the two parallel diameters l, m as above. Construct the point L in which l cuts the conic. Draw LL' perpendicular to l, and construct the point L' in which LL' cuts the conic again. Bisect LL' at V and draw f through V parallel to l. f is the axis. Construct the point A in which f cuts the conic. A is the vertex. Draw AZ perpendicular to f. Construct the tangent at a given point on the conic and let it cut AZ at Y. Then a perpendicular to this tangent at Y cuts f in the required focus.

6. If we are *given five points* on a conic, the conic can be constructed by Pascal's theorem (see XXIV. 2). If we are

given five tangents of the conic, the conic can be constructed by points (see XXIV. 7) or by tangents (see XXIV. 8).

Given four points and one tangent, to construct the conic.

Let $ABCD$ be the given points and t the given tangent. Let t cut the opposite sides of the quadrangle $ABCD$ in aa', bb', cc'. Take e, f, the double points of the involution (aa', bb', cc'). Then the two conics satisfying the required conditions are the conics through $ABCDe$ and through $ABCDf$. For let the conic through $ABCDe$ cut t again in e'. Then ee' belong to the involution (aa', bb', cc'), and e is a double point of this involution; hence e' coincides with e, i.e. t touches the conic through $ABCDe$. So it touches the conic through $ABCDf$.

7. *Given four tangents and one point, to construct the conic.*

Let OE, OF be the double lines of the involution subtended by the given quadrilateral at the given point O. Then the required conics are those touching the given lines and also touching OE or OF.

For the conic touching OE and the four given tangents, has, as tangents from O, OE, OE' where OE' is the ray corresponding to OE in the involution. But OE is a double ray. Hence OE and OE' coincide. Hence OE touches the conic at O. Hence the conic passes through O. So for the other conic.

Ex. 1. *Show that when four points are given and one tangent, the solution is unique if the line pass through one of the harmonic points.*

The other conic degenerates into a pair of opposite sides.

Ex. 2. *Show that there is no curved solution if the line pass through two harmonic points.*

Ex. 3. *Reciprocate Ex.* 1 *and Ex.* 2.

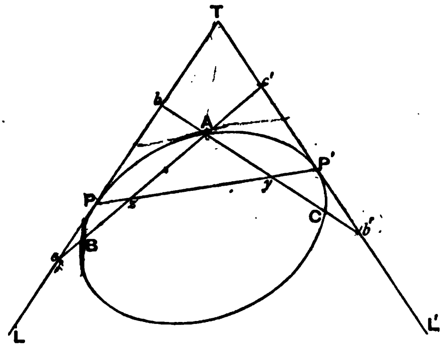
Ex. 4. *Describe a parabola through four given points.*

Ex. 5. *Construct a parabola, given three tangents and one point.*

***8.** *Given three points and two tangents, to construct the conic.*

Let the three points be A, B, C, and the two tangents

TL and TL'. Let AB cut TL and TL' in c and c', and let AC cut TL and TL' in b and b'. Take z, z', the double points of the involution (AB, cc'), and y, y' the double points of the involution (AC, bb'). Let any one, yz, of the four lines yz, yz', $y'z$, $y'z'$ cut TL and TL' in P and P'.



Then one conic satisfying the required conditions is the conic which passes through A and touches TL and TL' at P and P'. For let this conic cut AB again in B'. Then z is a double point of the involution (AB, cc') and also of the involution (AB', cc') (XXI. 5). Hence B and B' coincide, i.e. the conic passes through B. Similarly the conic passes through C.

So, by taking any of the lines yz', $y'z$, $y'z'$ instead of yz, we obtain another solution. Hence the problem has four solutions.

And there are no other solutions. For if AB cuts PP' at z, then z is a double point of the involution (AB, cc'). Hence z must be z or z'. So y must be y or y'.

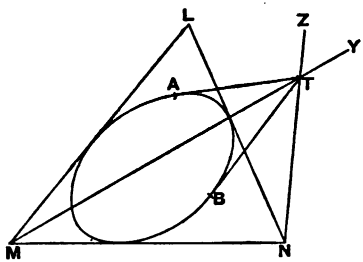
Note that since there are only four possible positions of the polar PP' of T, we have proved that—*If the sides BC, CA, AB of a triangle cut two lines TL and TL' in aa', bb', cc', and if the double points xx', yy', zz' of the involutions (BC, aa'), (CA, bb'), (AB, cc') be taken, then the six points $xx'yy'zz'$ lie three by three on four lines.*

***9.** *Given two points and three tangents, to construct the conic.*

Let A and B be the given points, and LM, MN, NL the given tangents. Take MY, MY' the double lines of the involution $M\,(AB, LN)$, and take NZ, NZ' the double

points of the involution $N(AB, LM)$. Let T be the meet of one of the lines MY, MY' with one of the lines NZ, NZ'. Describe a conic to touch TA and TB at A and B and to touch MN.

This is a conic satisfying the required conditions. For let ML' be the second tangent from M to this conic. Then MY is a double line of both the involutions $M(AB, NL)$ and $M(AB, NL')$ (XXIII. 4. ii). Hence ML' coincides with ML, i.e. the conic touches ML. So the conic touches NL.



By taking one of the other four meets instead of the meet of MY and NZ, we obtain three other solutions.

And there are no other solutions. For if the tangents at A and B meet at T, then MT is one of the double lines of the involution $M(AB, LN)$; so NT is one of the double lines of the involution $N(AB, LM)$.

10. *Given a triangle self-conjugate for a conic, and either two points on the conic, or one point on the conic and one tangent to the conic, or two tangents to the conic, to construct the conic.*

By V. 9, if we are given a self-conjugate triangle and one point, we are given three other points; and if we are given a self-conjugate triangle and one tangent, we are given three other tangents. In any of the above cases, therefore, the conic can now be constructed.

11. If we are given a focus, by XXVIII. 8 we are given two tangents. Hence the following problems belong to this chapter, but in each case a simpler solution can be given.

Given a focus and three points, to construct the conic.

Take the reciprocals of the given points for any circle with centre at the given focus, and draw a circle to touch these

lines. The reciprocal of this circle is the required conic. Since four circles can be drawn, there are four solutions.

Given a focus and two points and one tangent.

Reciprocation gives four solutions, two of which are imaginary.

Given a focus and one point and two tangents.

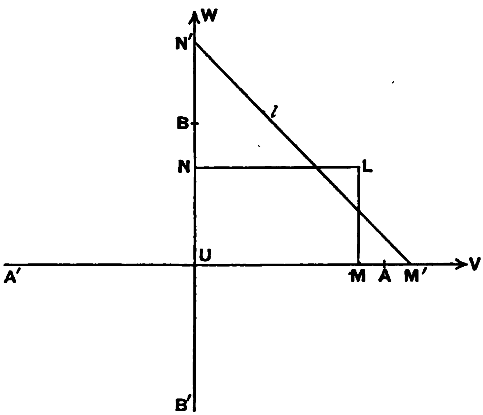
Reciprocation gives two solutions.

Given a focus and three tangents.

Reciprocation gives one solution. In this case, we can also solve the problem by determining the second focus by means of the theorem that two tangents to a conic are equally inclined to the focal radii to their meet.

12. *To construct a conic, given a self-conjugate triangle and a pole and polar.*

Let UVW be the given self-conjugate triangle, and L and l the given pole and polar. Project VW to infinity and



VUW into a right angle. Let l cut UV at M' and UW at N'. Draw LM parallel to UW to cut UV at M, and LN parallel to UV to cut UW at N. Take A and A' on UV such that $UA^2 = UA'^2 = UM \,.\, UM'$; and B and B' on UW

such that $UB^2 = UB'^2 = UN \,.\, UN'$. Describe a conic c' with axes AA' and BB'. Then if we project this conic back again, we have the required conic c.

c' is the only conic satisfying the required conditions in the new figure. For UV and UW must be the axes, being orthogonal conjugate diameters. Also the polar of M' (on l and UV) is LW, i.e. LM. Hence M and M' are conjugate points. Hence $UA^2 = UM \,.\, UM'$; so $UB^2 = UN \,.\, UN'$.

Also the conic c' satisfies the conditions of the problem. For since $UA^2 = UM \,.\, UM'$, the polar of M' passes through M; and also through W, and hence is LM. So the polar of N' is LN. Hence the polar of L is $M'N'$, i.e. l. Also UVW is a self-conjugate triangle since UV and UW are the axes. Hence in the case of c also the polar of L is l and UVW is a self-conjugate triangle.

The actual construction of the required conic is—Let WL cut UV at M, and let VL cut UW at N. Let l cut UV at M' and UW at N'. Take A, A' and B, B' the double points of the involutions (UV, MM') and (UW, NN'), and join WA. Then the required conic is the conic which passes through A', B, B' and touches WA at A. For since W is the pole of UV, WA touches the conic at A.

Note that since two sides of a self-conjugate triangle cut a conic in real points, the above construction is real if we choose those sides of the given triangle on which the involutions are non-overlapping.

Taking l at infinity, we obtain the solution of the problem—*Given the centre of a conic and a self-conjugate triangle, to construct the conic.*

M' and N' are now at infinity. Hence the points A, A' and B, B' on the conic are given by

$$MA^2 = MA'^2 = MU \,.\, MV \text{ and } NB^2 = NB'^2 = NU \,.\, NW.$$

We see again that *two conics cannot in general have two common self-conjugate triangles;* for since two such triangles more than determine a conic, the two conics would be coincident. For exceptional cases, see XI. 7.

Ex. 1. *Given a pentagon $ABCDE$, in which AB and CD meet at F, show that in the conic for which ADF is self-conjugate and E the pole of BC, the inscribed and circumscribed conics are reciprocal, each vertex being the reciprocal of the opposite side.*

Ex. 2. *Given a pole and polar and a self-conjugate triangle, construct the tangents from the pole.*

VL and WL cut l in points conjugate to N' and M'.

***13.** *Given five points on each of two conics, to construct the conic which passes through the (unknown) meets of these conics and also through a given point.*

Through the given point L draw any line l; and construct the points pp', qq' in which l cuts the two conics. Then if M be the other point in which the required conic cuts l, we know that pp', qq', LM are pairs in an involution. Hence M is known, i.e. a point on the conic is known on every line through L.

Given five points on each of two conics, to construct the conic which passes through the four unknown meets of these conics and also touches a given line.

Construct the points in which the given line cuts the given conics, viz. pp', qq'. Then the points of contact of the required conics are the double points e, f of the involution determined by pp', qq'. Then, taking either e or f, we continue as above.

Ex. *Give the reciprocal constructions.*

***14.** *Given three points on a conic and an involution of conjugate points on a line, to construct the conic.*

If the given involution has real double points, draw a conic through the three given points and the two double points. This conic clearly satisfies the required conditions.

If the given involution is overlapping, proceed thus—Let A, B, C be the given points, and l the line on which the involution of conjugate points lies. Let BC cut l in P, and take P', the mate of P, in the involution. Also take P''

such that $(BC, PP'') = -1$. Let PA cut $P'P''$ in A'', and take A' such that $(AA', PA'') = -1$. So, using CA and QQ', B' can be constructed.

Then the conic $ABCA'B'$ is the required conic. For since $(BC, PP'') = -1 = (AA', PA'')$, $P''A''$ is the polar of P. Hence PP' are conjugate points. So QQ' are conjugate points. Hence the involution (PP', QQ') (which is the given involution) is an involution of conjugate points for this conic.

If the given involution is overlapping, we have solved the problem—*To draw a conic through five given points, two of which are imaginary.*

***15.** *Construct a conic to pass through four given points and to divide a given segment harmonically.*

Let LM be the given segment. Let E, F be the double points of the involution determined by the given quadrangle $ABCD$ on LM. Let the double points P, Q of the involution (LM, EF) be constructed. Then the conic through $ABCDP$ is the required conic. For let LM cut this conic again in Q'. Then PQ' belong to the involution of the quadrangle on LM. Hence $(PQ', EF) = -1$. Hence Q' coincides with Q. And $(LM, PQ) = -1$. Hence the conic cuts LM harmonically.

If the double points E, F are imaginary, construct the involution of which L, M are the double points, and let P, Q be the common points of this involution and that of the quadrangle on LM. Then the required conic is $ABCDP$. For, as before, LM cuts the conic again in Q, and

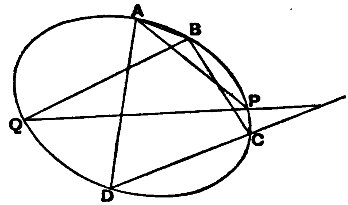
$$(LM, PQ) = -1.$$

Also, since E, F are imaginary, this construction is real.

Ex. *Construct a conic which shall pass through four given points and through a pair (not given) of points of a given involution on a line.*

16. The following proposition will be used in the succeeding constructions—

If a variable conic through four fixed points A, B, C, D meet fixed lines through A and B in P and Q, then PQ passes through a fixed point upon CD.



For consider the involution in which CD cuts the conic and the four sides AP, BQ, AB, PQ of the quadrangle $ABPQ$. Five of these points are fixed, viz. the meets with the fixed lines AB, AP, BQ, and the meets C, D with the conic. Hence the sixth meet is fixed, i.e. PQ passes through a fixed point on CD.

The theorem may also be stated thus—

A system of conics pass through $ABCD$. A fixed line through A cuts these conics in PP'..., and a fixed line through B cuts them in QQ'.... Then all the lines PQ, $P'Q'$,... are concurrent in a point on CD.

If A and B coincide, the theorem is—

A system of conics touch at A and pass through CD. A fixed line through A cuts these conics in P, P', ..., and another fixed line through A cuts them in Q, Q'.... Then all the lines PQ, $P'Q'$, ... are concurrent in a point on CD.

If A, B and C coincide, the theorem is—

A system of conics have three-point contact at A and pass through D. A fixed line through A cuts these conics in P, P', ..., and another fixed line through A cuts them in Q, Q', ... Then all the lines PQ, $P'Q'$, ... are concurrent in a point on AD.

If A, B, C, D coincide, the theorem is—

A system of conics have four-point contact at A. A fixed line through A cuts these conics in P, P',... and another fixed line through A cuts them in Q, Q', ... Then all the lines PQ, $P'Q'$, ... are concurrent on the common tangent at A.

Notice that, in the last three theorems, if the lines AP and AQ coincide, the lines PQ, $P'Q'$, ... become the tangents at P, P', ...

Ex. 1. *Reciprocate all these theorems.*

Ex. 2. *Given three meets ABC of two five-point conics, prove the following construction for the fourth meet D—Construct any two points L, M on either conic, and construct the points L', M' in which AL, BM cut the other conic. Join the meet of LM, $L'M'$ to C. Then D is the meet of this line with either conic.*

Ex. 3. *Given two meets A, B of two five-point conics, prove the following construction for the other meets C and D—Construct any two points L, M on either conic, and construct the points L', M' in which AL, BM cut the other conic. LM, $L'M'$ meet in one point on CD. Similarly construct another point on CD. Now construct the points in which the joining line cuts either conic.*

Ex. 4. *Reciprocate the two preceding constructions.*

***17.** *Given five points on a conic, three of which are coincident, to construct the conic.*

Let ABC be the three given coincident points, and DE the other given points. Then to be given ABC is equivalent to being given the point A, the tangent at A, and the circle of curvature at A. Let AD, AE cut this given circle in D', E'. Then DE, $D'E'$ meet on the common chord AP of the circle and the conic. Hence the point P where this chord cuts the circle can be constructed. Now P is on the conic. Hence we know four points A, D, E, P on the conic and the tangent at one of them. Hence the conic can be constructed.

Given five points on a conic, four of which are coincident; to construct the conic.

Let A, B, C, D be the four coincident points and E the fifth point. Then to be given A, B, C, D is to be given A, the tangent at A, the circle of curvature at A, and a conic c with which the conics have four-point contact at A. Let AE cut c at E' and let the tangent to c at E' cut the tangent to c at A at T. Now construct a conic having three-point contact with the given circle at A and passing through E and having TE as tangent at E. This is the required conic.

Ex. 1. *Obtain, by using the reciprocal theorem, a solution of the problem—Given five tangents of a conic, three of which are coincident, construct the conic.*

Notice that the circle of curvature has three-tangent contact with the conic as well as three-point contact.

Ex. 2. *Also of the problem—Given five tangents of a conic, four of which are coincident, construct the conic.*

XXIV AND XXV

1. Prove the following construction for the point P at which five given points A, B, C, D, E, no three of which are collinear, subtend a pencil homographic with a given pencil—Take the lines DD', DE' such that $D(ABCD'E')$ is homographic with the given pencil. Let the conic which passes through A, B, C and touches DD' at D cut DE' at F. Then FE cuts this conic at P.

2. Given four points on a conic and the direction of one asymptote, find the direction of the other.

3. Given three points on a conic and the directions of both asymptotes; construct the asymptotes.

4. Given a parallelogram, construct, using the ruler only, a parallel to a given line through a given point.

5. Verify the following construction (in which the ruler only is used) of a conic which shall pass through a given point and the (unknown) intersections of two conics, S and S', each given by five points—Draw any line PY through P, the given point. On PY construct the involution of conjugate points of S'. Let A be one of the given points on S. With A as vertex project this involution on to S. Let O be the pole of the involution on S. Find the polar of O with respect to S and let it intersect PY at X. Let AP intersect S at P_1. Join P_1X intersecting S again at Q_1. Then AQ_1 intersects PY at a point on the required conic.

6. Through a given point C draw a line cutting five given lines a, a', b, b', c' at five points A, A', B, B', C' such that (AA', BB', CC') may be an involution.

7. Given four points A, B, C, D and a line l. With A as pole of l and with BCD as a self-conjugate triangle the conic (A, BCD) is drawn. So the conics (B, CDA), (C, DAB) and (D, ABC) are drawn. Show that these four conics cut l in the same two points.

8. In the problem of constructing a conic, given a self-conjugate triangle and a pole and polar, show that if the pole lies on one of the sides of the triangle, the problem is either indeterminate or impossible.

CHAPTER XXVI

METHOD OF TRIAL AND ERROR

1. *Given two homographic ranges $(ABC\ldots)$ and $(abc\ldots)$ on different lines, and given two points V and v, find two points XY of the first range, such that the angles XVY and xvy may have given values, x and y being the points corresponding to X and Y in the homographic ranges.*

Try any point P on AB as a position of X. To do this, take Q on AB, so that the angle PVQ is equal to the given value of XVY. Take p and q, the points corresponding to P and Q. Also take r on ab, so that the angle pvr may be equal to the given angle xvy. Then if r coincides with q, the problem is solved.

If not, try several points P_1, $P_2\ldots$. Then

$$(r_1 r_2 \ldots) = v(r_1 r_2 \ldots)$$

$= v(p_1 p_2 \ldots)$ since the pencils are superposable

$= (p_1 p_2 \ldots) = (P_1 P_2 \ldots)$ since the ranges are homographic

$$= V(P_1 P_2 \ldots) = V(Q_1 Q_2 \ldots) = (Q_1 Q_2 \ldots) = (q_1 q_2 \ldots).$$

Hence the ranges $(q_1 q_2 \ldots)$ and $(r_1 r_2 \ldots)$ are homographic. Now if q and r coincide, q will be a position of y. Hence y is either of the common points of the homographic ranges $(q_1 q_2 \ldots)$ and $(r_1 r_2 \ldots)$. Hence Y and X are known.

The problem has four solutions. Two are obtained above, and two more are obtained by taking the angles PVQ and pvq in relatively opposite directions.

Notice that we need only make three attempts; for the common points of two homographic ranges can be determined if three pairs of corresponding points are known.

The above process may be abbreviated by writing (r) for the range $(r_1 r_2 \ldots)$, and so on.

The method is called by some writers *the method of False Positions.*

Ex. 1. *Find corresponding segments XY, $X'Y'$ of two given homographic ranges which shall be of given lengths.*

Ex. 2. *Given two homographic ranges on the same line, find a segment XX' bounded by corresponding points which divides a given segment harmonically.*

X and X' generate ranges which are in involution and therefore homographic.

Ex. 3. *Find also XX', given that XX' divides a given segment in a given cross ratio.*

Ex. 4. *Find corresponding points X, X' of two homographic ranges on different lines, such that XO and $X'O'$ meet at a given angle, O and O' being given points.*

The pencils at O and O' are homographic.

Ex. 5. *If A and A' generate homographic ranges on two lines, and B and B' generate homographic ranges on two other lines, find the positions of A, B, A', B' that both AB and $A'B'$ may pass through a given point.*

2. *Between two given lines, place a segment whose projections on two given lines shall be of given lengths.*

Let the projections lie on the lines AB and CD. On AB take a length LM equal to the given projection on AB; through L and M erect perpendiculars to AB to meet the given lines in X and Y. Let the projection of XY on CD be PQ. If PQ is of the required length, then the problem is solved.

If not, make PQ' of the required length. Then the ranges generated by Q' and P are homographic, being superposable. Again, the ranges P and X are homographic, by considering a vertex at infinity. Similarly

$$\text{range } X = \text{range } L = \text{range } M = \text{range } Y = \text{range } Q.$$

Hence the ranges Q' and Q are homographic. Either of the common points of these ranges gives a true position of Q.

Ex. 1. *On two given lines find points A and B, such that AB subtends given angles at two given points.*

Ex. 2. *Through a given point draw two lines, to cut off segments of given lengths from two given lines.*

Ex. 3. *Given two fixed points O and O' on two fixed lines, through a fixed point V draw a line cutting the fixed lines in points A, A', such that $OA : O'A'$ is constant.*

Ex. 4. *Through a given point draw a line to include with two given lines a given area.*

Here $OA \,.\, OA'$ is constant.

Ex. 5. *Given three rays OA, OB, OC, find three other rays OX, OY, OZ, such that the cross ratios $O(AB, XY)$, $O(BC, YZ)$, $O(CA, ZX)$ may have given values.*

Ex. 6. *Solve the equation $ax^2 + bx + c = 0$ by a geometrical construction.*

The roots are the values of x at the common points of the homographic ranges determined by $axx' + bx + c = 0$.

Ex. 7. *Solve geometrically the equations*

$$y = lx + a, \quad z = my + b, \quad x = nz + c.$$

Obtain the common points of the homographic ranges (x, x') determined by $y = lx + a$, $z = my + b$, $x' = nz + c$.

3. *Given two points L, M on a conic, find a point P on the conic, such that PL, PM shall divide the segment joining two given points U, V in a given cross ratio.*

Take any position of P, and let PL, PM meet UV in A, B, and take B' such that (UV, AB') is equal to the given cross ratio. Then $(A) = L(A) = L(P) = M(P) = M(B) = (B)$. Also, since (UV, AB') is constant, we have $(A) = (B')$. Hence $(B) = (B')$. Hence the required position of B is either of the common points of the homographic ranges generated by B and B'.

Ex. *Given two points L, M on a conic, find a point P on the conic such that the bisectors of the angle LPM may have given directions.*

Draw parallels to PL, PM through a fixed point.

4. *Inscribe in a given conic a polygon of a given number of sides, so that each side shall pass through a fixed point.*

Consider for brevity a four-sided figure. It will be found that the same solution applies to any polygon.

Suppose we have to inscribe in a conic a four-sided figure

$ABCD$, so that AB passes through the fixed point U, BC through V, CD through W, and DA through X. On the conic take any point A. Let AU cut the conic again in B. Let BV cut the conic again in C. Let CW cut the conic again in D. Let DX cut the conic again in A'. So take several positions of A.

Then the range on the conic generated by A is in involution with the range generated by B, since AB passes through a fixed point U. Hence $(A) = (B)$. So

$$(B) = (C) = (D) = (A').$$

Hence the ranges (A) and (A') on the conic are homographic. A true position of A is either of the common points of these homographic ranges.

Note that in the exceptional case of XXI. 3, Ex. 10, the common points lie on the line; and the above solution becomes nugatory.

Ex. 1. *Describe about a given conic a polygon such that each vertex shall lie on a given line.*

Inscribe in the conic a polygon whose sides pass through the poles of the given lines, and draw the tangents at its vertices; or use the reciprocal construction.

Ex. 2. *Inscribe in a given conic a polygon of a given number of sides, such that each pair of consecutive vertices shall determine with two given points on the conic a given cross ratio.*

Ex. 3. *In the given figure $ABCD$ inscribe the figure $NPQR$, so that RN, PQ meet in the fixed point U, and NP, RQ in the fixed point V.*

Ex. 4. *Construct a polygon, whose vertices shall lie on given lines and whose sides shall subtend given angles at given points.*

Ex. 5. *Construct a triangle ABC, such that A and B shall lie on given lines, and that the angle C shall be equal to a given angle, whilst the sides AB, BC, CA pass through fixed points.*

Ex. 6. *Given two homographic ranges $(ABC \ldots) = (A'B'C' \ldots)$ on a conic, find the corresponding points X, X', such that XX' may pass through a given point.*

Ex. 7. *Through four given points draw a conic which shall cut off from a given line a given length.*

XXVI

1. Solve geometrically the equations $xy+lx+my+n = 0$, $xy+px+qy+r = 0$.

2. Two sides of a triangle are given in position and the area is given. Show that the base in two positions subtends a given angle at a given point.

3. Given two pairs of conjugate diameters of a conic, construct a pair which shall include a given angle.

4. A ray of light starts from a given point, and is reflected successively from n given lines. Find the initial direction that the final direction may make a given angle with the initial direction.

5. Through a given point A is drawn a chord PQ of a conic. B, C are fixed points on the conic. Find the position of PQ when PB and QC meet at a given angle.

6. Through two given points describe a circle which shall cut a given circle in two points which make a given cross ratio with two given points on the circle.

7. Given two points A, B on the line of an involution, find a pair of points of the involution which shall divide AB harmonically.

CHAPTER XXVII

IMAGINARY POINTS AND LINES

1. The *Principle of Continuity* enables us to combine the elegance of geometrical methods with the generality of algebraical methods. For instance, if we wish to determine the points in which a line meets a circle, the neatest method is afforded by Pure Geometry. But in certain relative positions of the line and circle, the line does not cut the circle in real points.

Here Algebraical Geometry comes to our help. For if we solve the same problem by Algebraical Geometry, we shall ultimately have to solve a quadratic equation; and this quadratic equation will have two solutions, real, coincident or imaginary. Hence we conclude that a line always meets a circle in two points, real, coincident or imaginary.

Another instance is afforded by XXIII. 5. Here we prove the proposition by using the points O and O' in which the circles on AA' and BB' as diameters meet. But these circles in certain cases do not meet in real points. But we might have proved the same proposition by Algebraical Geometry, following the same method. Then it would have been immaterial whether the coordinates of the points O and O' had been real or imaginary, and the proof would have held good. Hence we conclude that we may use the imaginary points O and O' as if they were real.

In all solutions by Algebraical Geometry, points and lines will be determined by algebraical equations. Hence imaginary points and lines will occur in pairs. Hence we shall expect that in Pure Geometry, imaginary points and lines will occur in pairs.

2. The best way of defining the position of *a pair of imaginary points* is as the double points of a given overlapping involution; and the best way of defining the position of *a pair of imaginary lines* is as the double lines of a given overlapping involution.

Thus the points in which a line cuts a conic are the double points of the involution of conjugate points determined by the conic on the line; and these double points, i. e. the meets of the conic and line, are imaginary if the involution is an overlapping one.

So the tangents from any point to a conic are the double lines (real, coincident, or imaginary) of the involution of conjugate lines which the conic determines at the point.

Note that a pair of imaginary points is not the same as two imaginary points. For if AA' are a pair of imaginary points and BB' another pair of imaginary points, then AB are two imaginary points but are not a pair.

***3.** *The middle point of the segment joining a pair of imaginary points is real.*

For it is the centre of the involution defining the imaginary points.

A pair of imaginary points AA' is determined when we know the centre O and the square (a negative quantity) OA^2.

For the involution defining the points is given by

$$OP \,.\, OP' = OA^2.$$

The fourth harmonic of a real point for a pair of imaginary points is real.

For it is the corresponding point in the defining involution.

The product of the distances of a pair of imaginary points from any real point on the same line is real and positive.

Let AA' be the pair, and P any real point on the line AA'. Take O the middle point of the segment AA'. Then

$$PA \,.\, PA' = (OA - OP)(OA' - OP)$$
$$= (OA - OP)(-OA - OP) = OP^2 - OA^2.$$

Now OA^2 is negative, or the involution would have real double points. Hence $PA \cdot PA'$ is real and positive.

***4.** *Two conics cut in four real points, or in two real and two imaginary points, or in four imaginary points.*

Since a conic is determined by five points, two conics cannot cut in more than four points, unless they are coincident.

Also we can draw two conics cutting in four points, e.g. two equal ellipses laid across one another.

Now if we were solving the problem by Algebraical Geometry, and were given that the problem could not have more than four solutions, and that it had four solutions in certain cases, we should be sure that the problem had in all cases four solutions, the apparent deficiencies, if any, being accounted for by coincident or imaginary points.

Hence it follows by the Principle of Continuity, that two conics always cut in four points, real, coincident or imaginary.

Also imaginary points occur in pairs. Hence two or four of the points may be imaginary.

***5.** *If two conics cut in two real points, the line joining the other common points is real, even if the latter points are imaginary.*

For, by the principle of continuity, Desargues's theorem holds, even if two or four of the points on the conic are imaginary. Let any line cut the conics in pp' and qq' and the given real common chord in a. Then the real point a', taken such that (aa', pp', qq') is an involution, lies on the opposite common chord. Hence the opposite common chord is real, being the locus of the real point a'.

If two conics cut in two real and two imaginary points, one pair of common chords is real and two imaginary.

For if a second pair were real, the four common points would be real, being the meets of real lines.

***6.** *One vertex of the common self-conjugate triangle of two conics is always real.*

Take any line l; then the locus of the conjugate points of points on l for both conics is a conic. Take any other line m; the locus of the conjugate points of points on m for both conics is a second conic. These conics have one real point in common, viz. the conjugate point of the meet of l and m. Hence they have another real point in common, say U.

Take the conjugate point Q on l of U for both conics and the conjugate point R on m of U for both conics. Then QR is clearly the polar of U for both conics; for the polar of U for both conics passes through Q and R. Hence U is a real vertex of the common self-conjugate triangle of the two conics.

Similarly, the other two points, real or imaginary, in which the conics cut, are the other two vertices of the common self-conjugate triangle.

*7. *The other two vertices of the common self-conjugate triangle of two conics are real if the conics cut in four real points or four imaginary points; but if the conics cut in two real and two imaginary points, the other two vertices are imaginary.*

If the four intersections are real, the proposition is obviously true.

If the four intersections are imaginary, one conic must be entirely inside or entirely outside the other. Hence the polar of the real vertex U cuts the conics in either two non-overlapping segments AA', BB', or in one real segment and one imaginary, or in two imaginary segments. Now the other two vertices VW are the points on the polar which are conjugate for both conics, i. e. are the common pair of the two involutions of conjugate points on the polar. And the double points AA', BB' of these involutions are either real and non-overlapping, or one pair (at least) is imaginary. Hence by XX. 5, VW are real.

If two intersections are real and two imaginary, the meet of the given real common chord and of the opposite common chord (which is known to be real) gives a real position of U.

But the opposite chord does not cut either conic; hence U is outside both conics. Hence the polar of U, passing through the fourth harmonic of U for the two real points, cuts the two conics in overlapping real segments. Hence VW, being the double points of the involution determined by these segments, are imaginary.

***8.** *One pair of common chords of two conics is always real.*

If all four intersections are real, it is clear that the six common chords are all real.

If all four intersections are imaginary, then UVW are real. Take any point P and its conjugate point P' for the two conics. Then the common chords through U are the double lines of the involution $U(VW, PP')$; for the polar of P for these common chords passes through P', and the polar of V passes through W. Hence the common chords through U are both real or both imaginary.

Also the common chords through two of the three points UVW must be imaginary; for otherwise the four real common chords would intersect in four real common points of the conics. Let the chords through V and W be imaginary.

Then taking P inside the triangle UVW, we see that, since $V(UW, PP')$ overlap, P' must lie in the external angle V; so P' must lie in the external angle W. Hence P' lies in the internal angle U. Hence $U(VW, PP')$ does not overlap; hence the double lines of the involution are real, i. e. the common chords through U are real.

If two intersections are real and two imaginary, we have already proved that two common chords are real.

***9.** *Two conics have four common tangents, of which either two or four may be imaginary.*

If two conics have two real common tangents and two imaginary, the intersection of the real and also of the imaginary tangents is real; and the other four common apexes are imaginary.

One side of the common self-conjugate triangle of two conics is always real; the other two sides are real if the four common

tangents are all real or all imaginary; otherwise the other two sides are imaginary.

One pair of common apexes of two conics is always real.

These propositions can be proved similarly to the corresponding propositions respecting common points and common chords (or by Reciprocation).

Ex. *If two conics have three-point contact at a point, they have a fourth real common point, and a fourth real common tangent.*

CHAPTER XXVIII

CIRCULAR POINTS AND CIRCULAR LINES

1. The *circular lines* through any point are the double lines of the orthogonal involution at the point. Other names for circular lines are circulars or isotropic lines.

Notice that *a circular line is perpendicular to itself.*

Every pair of circular lines cuts the line at infinity in the same two points (called the *circular points* or circules or focoids).

Take any two points P and Q. Then to every pair of rays in the orthogonal involution at P there is a parallel pair of rays in the orthogonal involution at Q, or briefly, the involutions are parallel. Hence the double lines are parallel. Hence the circular lines through P and Q meet the line at infinity in the same two points.

The notation ∞, ∞' will be reserved for the circular points.

Any two perpendicular lines are harmonic with the circular lines through their meet.

For by definition the circular lines are the double lines of an involution of which the perpendicular lines are a pair.

The points in which any two perpendicular lines meet the line at infinity are harmonic with the circular points.

For the circular lines through the meet of the lines are harmonic with the given lines.

2. *The triangle whose vertices are any point C and the circular points, is self-conjugate for any rectangular hyperbola whose centre is at C.*

For $C\infty$, $C\infty'$ being circular lines are harmonic with every orthogonal pair of lines through C, and are therefore harmonic with the asymptotes, i. e. with the tangents from C to the r. h., and are therefore conjugate lines for the r. h.

Also C is the pole of $\infty\infty'$. Hence $C\infty\infty'$ is self-conjugate for the r. h.

Hence ∞, ∞' *are conjugate for any r. h.; and, conversely, if* ∞, ∞' *are conjugate for a conic, the conic is a r. h.* For if ∞, ∞' are conjugate, then $C\infty$, $C\infty'$ are conjugate diameters, and are therefore harmonic with the asymptotes, which are therefore orthogonal.

3. *All circles pass through the circular points.*

Let C be the centre of any circle. Then $C\infty$, $C\infty'$ are the asymptotes of the circle. For $C\infty$, $C\infty'$ are the double lines of the orthogonal involution at C, i. e. are the double lines of the involution of conjugate diameters of the circle. Now a conic passes through the points in which the line at infinity meets its asymptotes. Hence the circle passes through ∞ and ∞'.

Notice that we have proved that $C\infty$, $C\infty'$ touch at ∞, ∞' any circle whose centre is at C; for the asymptotes touch the conic at the points at infinity.

4. *Every conic which passes through the circular points is a circle.*

Let C be the centre of a conic through ∞, ∞'. Then since the lines joining the centre of a conic to the points where the conic meets the line at infinity are the asymptotes of the conic, we see that $C\infty$, $C\infty'$ are the asymptotes of the conic. Hence the involution of conjugate diameters of which the asymptotes are the double lines must be an orthogonal involution. Hence every pair of conjugate diameters is orthogonal. Hence the conic is a circle.

We now see the origin of the names circular points and circular lines. The circular points are the points through which all circles pass. A pair of circular lines is the limit of a circle when the radius is zero; the circle degenerating into a real point through which pass imaginary lines to the circular points. So that a pair of circular lines is both a circle and a pair of lines.

5. *Concentric circles have double contact, the line at infinity being the chord of contact.*

For all circles which have C as centre, touch $C\infty$ at ∞ and $C\infty'$ at ∞'.

Ex. 1. *Any semicircle APB is divided harmonically by the circular points.*

Join A, B, ∞, ∞' to P.

Ex. 2. *The circle which circumscribes a triangle which is self-conjugate for a rectangular hyperbola passes through the centre.*

For five of the vertices of the two triangles consisting of the given triangle and $C\infty\infty'$ lie on the circle.

Ex. 3. *Give a descriptive proof of the property of the director circle of a conic.*

Let A and B be any fixed points, and let PA and PB be any two lines through A and B which are conjugate for a conic. Draw the polar b of B cutting PA in Q. Then Q is the pole of PB. Hence $A(P) = A(Q) = (Q) = B(P)$. Hence the locus of P is a conic through A and B.

Now let R be any point on the director circle. Then $R\infty$, $R\infty'$ are conjugate for the conic, since the tangents from R, being perpendicular, are harmonic with $R\infty$, $R\infty'$. Hence the locus of R is a conic through ∞ and ∞', i. e. is a circle.

6. *If the pencil $V(ABC\ldots)$ be turned bodily through any angle about V into the position $V(A'B'C'\ldots)$, then the common lines of the two homographic pencils $V(ABC\ldots)$ and $V(A'B'C'\ldots)$ are the circular lines through V.*

The pencils, being superposable, are homographic. Hence if they cut any circle through V in $abc\ldots$ and $a'b'c'\ldots$, the two ranges $(abc\ldots)$ and $(a'b'c'\ldots)$ on the circle are homographic. One point on the homographic axis of these ranges is the meet of ab' and $a'b$. But these lines are parallel; since the angles aVa', bVb' and therefore the arcs aa', bb' are equal. Hence this point is at infinity. So every point on the axis is at infinity. Hence the common points of the ranges $(abc\ldots)$ and $(a'b'c'\ldots)$ are the meets of the circle with the line at infinity, i. e. are ∞, ∞'. Hence the common lines of the pencils $V(ABC\ldots)$ and $V(A'B'C'\ldots)$ are $V\infty$, $V\infty'$.

Conversely, *if $V\infty$, $V\infty'$ are the common lines of the homographic pencils $V(ABC\ldots)$ and $V(A'B'C'\ldots)$, then*

$$AVA' = BVB' = CVC' = \ldots.$$

For, if not, let $AVA' = BVB'' = CVC'' = \ldots$. Then by the first part $V(\infty\infty' ABC \ldots) = V(\infty\infty' A'B''C'' \ldots)$. But by hypothesis $V(\infty\infty' ABC \ldots) = V(\infty\infty' A'B'C' \ldots)$. Hence VB'' and VB' coincide, each being the ray corresponding to VB in the homography determined by $V(\infty\infty' A) = V(\infty\infty' A')$. Hence $BVB' = BVB'' = AVA'$. So for the other angles.

The legs of a constant angle divide the segment joining the circular points in a constant cross ratio.

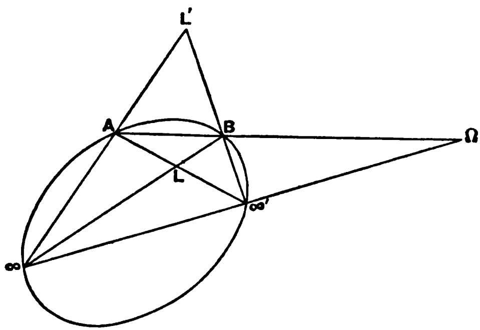
Let the constant angles be ALA', BMB', CNC', ... Through any point V draw a circle and let parallels through V to LA, MB, NC, ..., LA', MB', NC' ... cut this circle in $a, b, c, \ldots, a', b', c' \ldots$. Then, as above, $\infty\infty'$ are the common points of the homographic ranges $(abc \ldots)$ and $(a'b'c' \ldots)$ on the circle. Hence by XVI. 4. ii,

$$(\infty\infty', aa') = (\infty\infty', bb') = (\infty\infty', cc') = \ldots.$$

Hence $V(\infty\infty', aa')$ is constant. But the parallel lines LA and Va cut $\infty\infty'$ in the same point; so LA' and Va' cut $\infty\infty'$ in the same point. Hence $L(\infty\infty', AA')$ is constant. Hence LA and LA' divide the segment $\infty\infty'$ in a constant cross ratio.

7. *Coaxal circles are a system of four-point conics.*

For two circles meet in two points (real or imaginary)



on the radical axis and also in the circular points. The adjoining ideal figure explains the relation of coaxal circles to the circular points. A and B are the common finite

points on the radical axis, and Ω is the point at infinity on the radical axis.

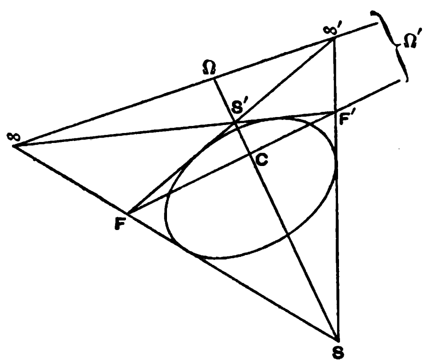
L and L' are the limiting points. For since LA and LB are circular lines through A and B, L is a point-circle of the system. So for L'. Also $LL'\Omega$ is the common self-conjugate triangle of the coaxal system.

Foci of a Conic.

8. *Every conic has four foci, which are inside the conic and lie two on each axis, those on either axis being equidistant from the centre.*

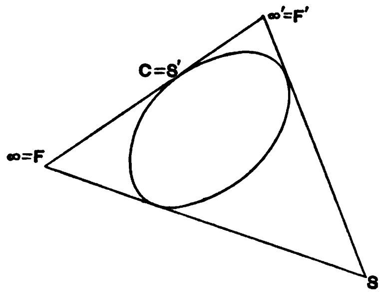
The tangents from a focus of a conic to the conic are the double lines of the involution of conjugate lines at the focus, i. e. are the double lines of an orthogonal involution, i. e. are circular lines, i. e. pass through ∞, $'$. Hence a real focus is an internal point, since the tangents from it are imaginary.

Also every intersection S of the four tangents from ∞, ∞' to the conic is a focus of the conic. For $S\infty$, $S\infty'$ being the tangents from S and also circular lines, the involution of conjugate lines at S is orthogonal, i. e. S is a focus. Hence the foci of a conic are the other four meets of tangents to the conic from ∞ and ∞'.

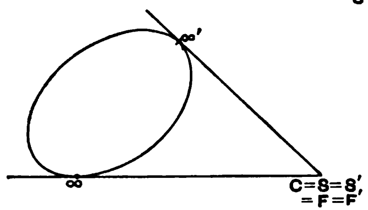


Consider the adjoining ideal figure. Here $SS'FF'$ are the foci. Also $C\Omega\Omega'$ is a self-conjugate triangle, being the harmonic triangle of a circumscribing quadrilateral. Hence C, being the pole of $\infty\infty'$, is the centre. Again $(\infty\infty', \Omega\Omega')$ is harmonic. Hence $C(\infty\infty', \Omega\Omega')$ is harmonic. Hence the conjugate diameters $C\Omega$, $C\Omega'$, being harmonic with the circular lines $C\infty$, $C\infty'$,

are orthogonal. Hence SS' and FF' are the axes; i.e. the foci lie two on each axis. But $(SS', C\Omega)$ is harmonic and Ω is at infinity; hence S and S' are equidistant from the centre C. Similarly for F and F'.



It will be instructive to draw an ideal picture showing the relation of a parabola and of a circle to its foci.



In the case of a parabola $\infty\infty'$ touches the conic. Hence F' coincides with ∞' and F with ∞. Also C and S' coincide at the point of contact of $\infty\infty'$.

In the case of a circle, ∞ and ∞' are on the conic; and all the foci coincide with the centre C.

Ex. 1. *The sides of a triangle ABC touch a conic and meet a fourth tangent to the conic in $A'B'C'$; show that the double lines of the involution subtended by (AA', BB', CC') at a focus are perpendicular.*

For $S\infty$, $S\infty'$, being the tangents from S, belong to this involution.

Ex. 2. *The circle described about a triangle which circumscribes a parabola, passes through the focus.*

For five of the vertices of the two triangles consisting of the given triangle and $S\infty\infty'$ lie on the circle.

Ex. 3. *The intersections of a conic with its directrices are points on the director.*

Let P be such a point. Then P is a point on a polar of a focus, say, the point of contact of $S\infty$. Now the tangents from P (a point on the conic) coincide with $P\infty$, a circular line, which is perpendicular to itself. Hence P is on the director.

9. *The foci on one axis* (called the *focal axis*) *are real, and the foci on the other axis* (called the *non-focal axis*) *are imaginary.*

Take any point P, and through P draw the orthogonal pair PGg and PHh of the involution of conjugate lines at P, cutting one axis in G and H and the other axis in g and h. Then PG and PH are harmonic with $P\infty$ and $P\infty'$, since GPH is a right angle, and with the tangents PT, PT' from P, since PG and PH are conjugate. Hence PG and PH are the double lines of the involution $P(TT', \infty\infty', SS', FF')$.

Hence $P(SS', GH)$ and $P(FF', gh)$ are harmonic. And C bisects SS' and FF'. Hence $CS^2 = CG \cdot CH$ and

$$CF^2 = Cg \cdot Ch.$$

But on drawing the figure, we see that if CG and CH are of the same sign, Cg and Ch are of opposite signs. Hence, taking $CG \cdot CH$ positive, CS^2 is positive and CF^2 is negative. Hence S and S' are real and F and F' are imaginary.

Notice that *in the case of a parabola, S bisects GH*; for S' is at infinity.

Ex. 1. *Show that gh subtends a right angle at S and at S'.*

Now $Cg \cdot Ch = -CG \cdot CH$ by elementary geometry $= -CS^2 = CS \cdot CS'$. Hence $SS'gh$ lie on the circle whose diameter is gh.

Ex. 2. *Any line through G is conjugate to the perpendicular line through H; and the same is true of g and h.*

10. *Confocal conics are a system of four-tangent conics.*

For if S and S' be the real foci, the conics all touch the lines $S\infty$, $S'\infty$, $S\infty'$, and $S'\infty'$.

Hence, *the tangents from any point to a system of confocals form an involution, to which belong the pairs* (PS, PS'), (PF, PF') *and* $(P\infty, P\infty')$, *P being the given point.*

Through every point can be drawn a pair of lines which are conjugate for every one of a system of confocals.

For the double lines PG, PH of the above involution are harmonic with the tangents from P to each of these conics.

PG and PH are perpendicular.

For they are harmonic with $P\infty$, $P\infty'$.

The pairs of tangents from any point to a system of confocals and the focal radii to the point have a common pair of bisectors.

For the double lines PG and PH of the involution are perpendicular.

Notice that *in the case of a parabola, PG and PH are the bisectors of the angles between PS and a parallel through P to the axis.*

Ex. 1. *From a given point O, lines are drawn to touch one of a system of confocal conics in P and Q; show that PQ and the normals at P and Q touch a fixed parabola which touches the axes of the confocals.*

Viz. the polar-envelope of the point O for the system of four-tangent conics. The normal PG at P touches the polar-envelope, because it is conjugate to OP for every conic of the system. Also $\infty\infty'$ and the axes touch, since they are the harmonic lines of the quadrilateral.

Ex. 2. *The directrix of the parabola is CO, C being the common centre.*

For the tangents at O to the two confocals through O are two positions of PQ.

11. *The locus of the poles of a given line for a system of confocals is the normal at the point of contact of the given line with a confocal.*

For let the given line l touch a confocal at P, and let PG be the normal, and $PH (= l)$ the tangent to this confocal. Then PG and PH are perpendicular. Hence $P(GH, \infty\infty')$ is harmonic. But PH is one of the double lines of the involution of tangents from P to the confocals, being the pair of coincident tangents from P to the confocal which PH touches. And $P\infty$, $P\infty'$ is a pair of this involution. Hence PG is the other double line. Hence PG and PH, being harmonic with every pair of tangents, are conjugate for every confocal. Hence the locus of the poles of l is PG.

Reciprocation of circular points and lines.

***12.** Circular lines are the double lines of the orthogonal involution at a point P. Hence *the reciprocal of a pair of circular lines* is a pair of points on a line p which are the double points of the involution on the line which subtends an orthogonal involution at the origin O of reciprocation, in other words, are the meets of p with the circular lines through the origin of reciprocation.

Circular points are the points on the line at infinity which are the double points of the involution on the line at infinity which subtends an orthogonal involution at O. Hence *the reciprocals of the circular points* are the double lines of the orthogonal involution at O, i. e. are the circular lines through the origin of reciprocation.

The reciprocal of a circle for the point O is a conic with focus at O.

For since the circle passes through the circular points, the reciprocal touches the circular lines through O, i. e. O is a focus of the reciprocal.

To reciprocate confocal conics into coaxal circles.

Confocal conics are conics inscribed in the quadrilateral $S\infty, S'\infty, S\infty', S'\infty'$. Reciprocate for S. Then since $S\infty$, $S\infty'$ touch the given conics, the circular points lie on the reciprocal conics, i. e. the reciprocal conics are circles. Also the given conics have two other common tangents; hence the reciprocal conics have two other common points, i. e. are coaxal circles.

To reciprocate coaxal circles into confocal conics.

Coaxal circles are conics circumscribed to the quadrangle $AB\infty\infty'$. (See figure of § 7.) Reciprocate for L. Then the given conics pass through four fixed points, two on each circular line through the origin of reciprocation. Hence the reciprocal conics touch four fixed lines, two through each of the circular points; i. e. the tangents to all the reciprocal conics from ∞, ∞' are the same, i. e. the reciprocal conics are confocal.

XXVIII

1. A circle is drawn with centre on the directrix of a parabola to pass through the focus. At R, one of the intersections of the parabola and the circle, are drawn the tangents to the circle and the parabola, meeting the parabola and the circle again at P and Q. Show that the circle and the parabola are manifoldly related; and deduce that PQ touches both curves.

2. OP and OQ are the tangents from a fixed point O to one of a system of confocal conics. Show that the circles such as OPQ are coaxal.

3. OP and OQ are the tangents from a fixed point O to one of a system of confocal conics, and the normals at P and Q meet at R. Show that the locus of the orthocentre of PQR is a line.

4. Every conic through the foci of a conic is a rectangular hyperbola.

5. If the two parabolas which can be drawn through four given points have their axes perpendicular, the points lie on a circle.

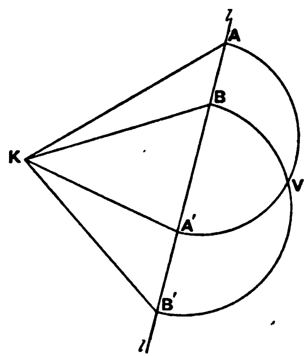
CHAPTER XXIX

PROJECTION, REAL AND IMAGINARY

1. *To project a given conic into a circle and at the same time a given line to infinity.*

Take K, the pole of the given line l which is to be projected to infinity. Through K draw two pairs of conjugate lines cutting l in AA', BB'.

On AA' and BB' as diameters describe circles cutting in V and V'. About AA' rotate V out of the plane of the paper. With V as vertex project the given figure on to any plane parallel to the plane VAA'.



Then KA will be projected into a line parallel to VA, and KA' into a line parallel to VA'. Hence AKA' will be projected into a right angle. So BKB' will be projected into a right angle. Again, since KA and KA' are conjugate for the given conic, their projections will be conjugate for the conic which is the projection of the given conic. So KB and KB' will be conjugate in the figure obtained by projection. Again, K is the pole for the given conic of the given line l which is projected to infinity. Hence in the second figure, K is the pole of the line at infinity, i.e. is the centre of the conic.

Hence in the second figure KA, KA' and KB, KB' are two pairs of orthogonal conjugate lines at the centre, i.e. the second conic has two pairs of orthogonal conjugate diameters. Hence the second conic is a circle.

If the conic is given by five points, we should start in the above construction with the point A on l and construct by XXIV. 2, the polar KA' of this point. Then KA' is the line conjugate to KA. So KB and KB' can be constructed.

Notice that we have thus solved the problem—*Project any five points so that their projections shall lie on a circle whose centre shall be the projection of a given point.*

2. The above construction fails when the line to be projected to infinity is the line at infinity itself. The problem then becomes—

To project a given conic into a circle, so that the centre of the conic may be projected into the centre of the circle.

If the conic is an ellipse, this can be done at once by Orthogonal Projection. If the conic is a hyperbola, we must use an imaginary Orthogonal Projection. If the conic is a parabola, the projection is impossible; but we may take the parabola as the limit of an ellipse with one vertex at infinity.

Notice that, as the line at infinity is projected into the line at infinity in Orthogonal (or Parallel) Projection, the species of the conic is not altered by this kind of Projection. For if the line at infinity cuts the conic in real, coincident, or imaginary points in the original figure, the line at infinity in the new figure will also cut the conic in real, coincident, or imaginary points respectively.

Ex. *Project a system of homothetic conics into circles.*

Project any one into a circle.

3. *To project a given conic into a circle and a given point into its centre.*

In the above construction, take K to be the given point and l its polar.

To project a given conic, so that one given point may be projected into the centre and another given point into a focus.

To project L into the centre and K into a focus, take l in

the above construction to be the polar of L instead of the polar of K, using K and l as before. Then L is projected into the pole of the line at infinity, i.e. into the centre, and K is projected into a point at which two pairs of conjugate lines are orthogonal, i.e. into a focus.

To project a given conic, so that two given points may be projected into its foci.

To project K, K' into the foci, take L and L', the double points of the involution (PP', KK'), P and P' being the points in which KK' cuts the conic. Now project K into a focus and L into the centre. Then (KK', LL') is harmonic; also L' is at infinity, for since (PP', LL') is harmonic, L' is on the polar of L. Hence KK' is bisected at L, i.e. K' is the other focus.

Ex. 1. *Project a given conic in a given plane into a circle in another given plane.*

In the construction of § 1, take the line AA' parallel to the intersection of the two planes, and take V in the plane through AA' parallel to the second plane.

Ex. 2. *Project a given conic into a parabola, and a given point into its focus, and a given point on the conic into the vertex of the parabola.*

Suppose we want to project S into the focus, and P into the vertex of a parabola. Let SP cut the conic again in P'. Taking the tangent at P' as vanishing line, project S into a focus.

4. In the fundamental construction of § 1, if the point K be outside the conic, the pencil of conjugate lines at K is not overlapping; hence the segments AA', BB' do not overlap; hence the points V and V' are imaginary. In this case we say that the vertex of projection is imaginary, and that we can by an *imaginary projection* still project the conic into a circle and l to infinity. Also, by the Principle of Continuity, proofs which require an imaginary projection are valid; in fact we need not pause to inquire whether the projection is real or whether it is imaginary.

The reader will find that the method of Projection (combined if necessary with the method of Reciprocation)

is very powerful; and he should try to solve in this way any problem which he cannot do in other ways. Many propositions can also be proved most easily in this way. For instance—

Prove Pascal's theorem by projection.

(See figure of XV. 1.) Project MN to infinity and the conic into a circle. Then, in the circle, we have AB parallel to DE, and BC parallel to EF. Hence

$$FAD = 180° - FED = 180° - ABC \text{ by parallels} = ADC.$$

Hence AF is parallel to CD; i.e. AF and CD meet on the line at infinity, MN. Hence in the original figure AF and CD meet on MN.

Ex. 1. *Prove by Projection that the harmonic triangle* (i) *of an inscribed quadrangle,* (ii) *of a circumscribed quadrilateral, is self-conjugate for the conic.*

Project in each case into a parallelogram, and notice that a parallelogram inscribed in a circle must be a rectangle.

Ex. 2. *Show that the harmonic triangles of a quadrangle inscribed in a conic and of the quadrilateral of tangents at the vertices of the quadrangle are coincident.*

Ex. 3. *The chords PP', QQ', RR', SS' of a conic meet in O. Show that the two conics $OPQRS$ and $OP'Q'R'S'$ touch at O.*

Project the conic into a circle and O into its centre. Then the two conics are the reflexions of one another in O. Hence the tangents at O coincide.

Ex. 4. *If two coaxal triangles be inscribed in* (*or circumscribed to*) *a conic, the c. of h. is the pole of the a. of h.*

Ex. 5. *The lines joining the vertices of a triangle ABC inscribed in a conic to a point O meet the conic again in a, b, c; and Ab, Bc, Ca meet the polar of O in R, P, Q. Show that the lines joining any point on the conic to P, Q, R meet BC, CA, AB in collinear points.*

Ex. 6. *If from three collinear points X, Y, Z, pairs of tangents be drawn to a conic, and if ABC be the triangle formed by one tangent from each pair, and DEF the points in which the remaining three tangents meet any seventh tangent, the lines AD, BE, CF meet at a point on XYZ.*

Let the tangents from X be called l, l', from Y be called m, m', and from Z be called n, n', and the line XYZ be

called o, and the seventh tangent be called p. Reciprocating, we have to prove the theorem—'If LOL', MOM', NON' be chords of a conic, and P any point on the conic, then the meets of LM, PN', of MN, PL', and of NL, PM' lie on a line through O.' Project to infinity the line joining O to the meet of LM, PN', and at the same time the conic into a circle. The theorem becomes—'If LL', MM', NN' be parallel chords of a circle and P a point on the circle such that PN' is parallel to LM, then PM' is parallel to NL and PL' to MN.' This theorem follows by elementary geometry.

Ex. 7. *ABC is a triangle inscribed in a conic of which O is the centre. A', B', C' bisect BC, CA, AB. Through P, any point on the conic, are drawn lines parallel to OA', OB', OC' meeting BC, CA, AB in X, Y, Z; show that X, Y, Z are collinear.*

By an Orthogonal Projection, real or imaginary, project the given conic into a circle with O as centre. Then in the circle, OA' is perpendicular to BC, OB' to CA, and OC' to AB.

Ex. 8. *Through a fixed point O is drawn a chord PP' of a conic; show that the locus of the middle point of PP' is a homothetic conic through O and through the points of contact of tangents from O.*

Ex. 9. Hesse's theorem. *AA', BB', CC' are the opposite vertices of a complete quadrilateral. Show that if AA' and BB' are two pairs of conjugate points for a conic, then C, C' are also conjugate points for this conic.* (See also XIV. 3, Ex. 3.)

Project the line AA' to infinity and the conic into a circle. Then since A, A' are points at infinity which are conjugate for a circle, they subtend a right angle at the centre and therefore at any point. Hence the parallelogram $BB'CC'$ is a rectangle. Now B, B' are conjugate for the circle. Hence the circle on BB' as diameter is orthogonal to the circle. But the circle on BB' as diameter is also the circle on CC' as diameter. Hence the latter circle is also orthogonal to the circle. Hence C, C' are conjugate for the circle. Hence in the original figure C, C' are conjugate for the conic.

Ex. 10. *Prove, by Projection, the involution property of a point and a conic; viz. that if chords be drawn through a fixed point O cutting the conic at the pairs of points PP', QQ', RR', ...,*

then $(PP', QQ', RR', \dots)$ *form an involution on the conic.* (See also XX. 2.)

Project the conic into a circle and O into the centre of the circle. Then we have to prove that $(PP', QQ', RR', \dots)$ is an involution on the circle. We see that $P', Q', R', \dots$ are the reflexions of $P, Q, R, \dots$ in the centre O. Take V and V' at the ends of any diameter. Now we have to prove that $(PP'QQ'RR'\dots) = (P'PQ'QR'R\dots)$; which is true if $V(PP'QQ'RR'\dots) = V'(P'PQ'QR'R\dots)$. But these pencils are superposable, being the reflexions of one another in the centre O of the circle. Hence $(PP', QQ', RR', \dots)$ form an involution in the figure of the circle, and therefore in that of the conic.

Ex. 11. *If two conics are such that one four-sided and four-angled figure can be drawn to be inscribed in one conic and circumscribed to the other, then an infinite number of such figures can be drawn.*

Let $ABCD$ be the figure which is inscribed in c_1 and circumscribed to c_2; and let AC and BD meet at U. Project c_1 into a circle, c_1', and U into its centre. Then since $B'D'$ is a diameter of c_1', $B'A'D' = 90°$; hence A' is on the director of c_2', the projection of c_2. So B', C', D' are on this director. Hence c_1' is this director. Now start with any point P' on c_1', and let the tangents from P' to c_2' cut c_1' at Q' and S', and let the other tangents from Q' and S' to c_2' intersect at R'. Then $P'Q'R'S'$ is a rectangle, since the angles at P', Q', S' are right. Hence R' also lies on the director. Hence, in the original figure, we can draw such a figure starting with any point P on c_1.

Ex. 12. *Show that the internal diagonals of such a figure intersect at a fixed point which is one of the vertices of the common self-conjugate triangle of the conics, and that the external diagonal is a fixed line, viz. the polar of the above point for each conic.*

Ex. 13. *If any circle be drawn through the real foci of a conic, quadrilateral figures can be inscribed in the circle which are circumscribed to the conic.*

5. *To project any two given imaginary points into the circular points.*

Let the two imaginary points E, F be given as the double points of the overlapping involution (AA', BB'). Take any point K in the given plane and proceed as in § 1 to project

the angles AKA' and BKB' into right angles and AA' to infinity. Then KE and KF are the double lines of the orthogonal involution $K(AA', BB')$, and E and F are at infinity; hence E and F are the circular points.

To project any two imaginary lines into a pair of circular lines.

Let the given lines KE, KF be defined as the double lines of the involution $K(AA', BB')$. Draw any transversal $AA'BB'$. Then proceed as in § 1 to project the angles AKA' and BKB' into right angles. Then KE and KF, being the double lines of an orthogonal involution, are circular lines.

If the points or lines are real instead of imaginary, the projection becomes imaginary.

To project any conic into a rectangular hyperbola.

Project any two conjugate points into the circular points.

To project a system of angles which cut a given line in two homographic ranges, into equal angles.

Project the common points into the circular points.

Ex. 1. *Deduce the construction for drawing a conic to touch three lines and to pass through two points from the construction for drawing a circle to touch three lines.*

Ex. 2. *The pole-locus of four given points A, B, C, D and a given line l, touches the sixteen conics which can be drawn through the two common conjugate points on l to touch the sides of one of the triangles ABC, ACD, ADB, BCD.*

Project these conjugate points into the circular points; then l goes to infinity. Also AD, BC meet the line at infinity in points harmonic with the circular points; hence AD, BC are perpendicular. Similarly BD, AC are perpendicular, and also CD, AB. Hence the pole-locus becomes the centre-locus of rectangular hyperbolas through A, B, C, D, i.e. becomes the nine-point circle of each of the four triangles; and this is known to touch any circle which touches the sides of any one of the four triangles.

Ex. 3. *The lines AB, AC touch a conic at B and C; and the lines PQ and PR touch the conic at Q and R. Show that A, B, C, P, Q, R lie on a conic. Through A is drawn a line cutting the given conic at L and M and cutting QR at N, and*

a point U is taken such that (LM, NU) is harmonic. Show that U lies on the second conic.

Project Q, R into the circular points. Then the conic becomes a circle with centre P, and U becomes the middle point of LM. And now a circle goes through $ABUPC$. Hence, in the original figure, a conic goes through $ABUPCQR$.

Ex. 4. *Show that in Ex. 3 the conic through A, B, C, P, Q, R is also the locus of a point X, from which tangents to the conic are harmonic with XP and XA.*

6. *To project any two conics into circles.*

Project any two common points into the circular points, *or* project one conic into a circle and a common chord to infinity.

There are six solutions, as there are six common chords. But the projection is only real if we take a real common chord which meets the conics in imaginary points, for the line at infinity satisfies these conditions.

To project a system of four-point conics into a system of coaxal circles.

Proceed as above.

Ex. 1. *Given two tangents and two points on a conic, the locus of the meet of the tangents at these points is two lines.*

Project the two points into the circular points.

Ex. 2. *Two conics pass through $ABCD$. AEF, BGH cut the conics in EG, FH; show that CD, EG, FH are concurrent.*

Ex. 3. *A variable conic passing through four fixed points A, B, C, D meets a fixed conic through AB in PQ; show that PQ passes through a fixed point.*

Ex. 4. *If a conic pass through two given points and touch a given conic at a given point, its chord of intersection with the given conic passes through a fixed point.*

Ex. 5. *The poles of a common chord of two conics are divided harmonically by the two common apexes* (see XIX. 8) *which lie on the line joining the poles.*

Project the common points into the circular points, and notice that S, S' divide the line joining the centres of the circles harmonically.

Ex. 6. *If tangents be drawn from any point on any common*

chord of two conics, touching one conic in A, B and the other in C, D; show that the lines AC, AD, BC, BD meet two by two in the common apexes corresponding to the common chord.

Ex. 7. *If through any common apex of two conics a line be drawn cutting the conics in the points AB and CD, at which the tangents are ab and cd; show that the points ac, ad, bc, bd lie two by two on the corresponding common chords.*

Ex. 8. *If three conics have two points in common, the opposite common chords of the conics taken in pairs, are concurrent.*

Ex. 9. *The envelope of a line which meets two given conics in pairs of harmonic points is a conic (called the harmonic envelope of the two conics) which touches the eight tangents to the conics at their meets.*

Let the conics meet in $ABCD$. Project AB into the circular points. Then by Ex. 2 of III. 6, the envelope of the line is a conic which touches the four tangents at C and D. So by projecting CD into the circular points, we prove that the envelope touches the tangents at A and B.

Ex. 10. *The locus of a point the tangents from which to two given conics are pairs of a harmonic pencil is a conic (called the harmonic locus of the two conics) on which lie the eight points in which the given conics touch their common tangents.*

Reciprocate Ex. 9.

Ex. 11. *The tangents to a system of four-point conics at their meets form four homographic pencils.*

Ex. 12. *Reciprocate Ex.* 11.

Ex. 13. *If two conics be so situated that two of their meets AB subtend at another meet C an angle which divides harmonically the tangents at C, the same is true for AB at D, for CD at A, and for CD at B.*

Apply Ex. 11 to the four conics consisting of the two given conics and the pair of lines AC, BD and the pair AD, BC.

Ex. 14. *Four parabolas are drawn with their axes in the same direction to touch the four triangles formed by four points; show that they have a common tangent.*

A particular case of the more general theorem—'Four conics are drawn to touch two given lines and to touch, &c.'

Reciprocate, and project the given points into the circular points.

Ex. 15. *Prove the involution property of a system of four-point conics.*

See the figure of XXI. 2. Project C, D into the circular points. Then the system becomes a system of coaxal circles with radical axis AB. Hence $c'p \cdot c'p' = c'q \cdot c'q' = \dots$. Hence pp', qq', ... form an involution of which c' is the centre. But c is at infinity. Hence cc' are a pair in the involution. So aa', bb' are pairs in the involution.

7. *To project any two conics into confocal conics.*

Let the opposite vertices of the quadrilateral circumscribed to both conics be AA', BB', CC'. Project AA' into the circular points; then the conics have the foci BB', CC' in common, i.e. are confocal.

To project a system of conics inscribed in the same quadrilateral into confocal conics.

Project a pair of opposite vertices of the circumscribing quadrilateral into the circular points.

Ex. 1. *A variable conic touches four fixed lines; from the fixed points B, C taken on two of these lines the other tangents are drawn; find the locus of their meet.*

Project B, C into the circular points.

Ex. 2. *The line PQ touches a conic. Find the locus of the meet of tangents of the conic which divide PQ* (i) *harmonically,* (ii) *in a constant cross ratio.*

Ex. 3. *If two conics be inscribed in the same quadrilateral, the two tangents at any of their meets cut any diagonal of the quadrilateral harmonically.*

Ex. 4. *Given the cross ratio of a pencil, three of whose rays pass through fixed points and whose vertex moves along a fixed line, the envelope of the fourth ray is a conic touching the three sides of the triangle formed by the given points.*

***8.** *To project any two conics into homothetic conics.*

Project any common chord to infinity. The new conics pass through the same two points at infinity, and hence are homothetic. (See XIX. 11, end.)

To project any two conics which have double contact into homothetic and concentric conics.

Project the chord of contact to infinity. The pole of the chord of contact projects into the common centre.

9. *To project any two conics having double contact into concentric circles.*

Project the two points of contact into the circular points. Then the conics will both pass through the circular points, i.e. will both be circles. Also they will both have the same pole of the line at infinity, i.e. they will be concentric.

Ex. 1. *Conics having the same focus and corresponding directrix can be projected into concentric circles.*

For the focus S has the same polar, and the tangents from S are the same. Hence the conics have double contact.

Ex. 2. *Through the fixed point O is drawn a chord OAB of a conic, and on OAB is taken the point P such that $(OABP)$ is constant. Show that the locus of P is a conic having double contact with the given conic.*

Project the conic into a circle and O into its centre.

10. *The lines which join pairs of corresponding points of two homographic ranges on a conic, touch a conic having double contact with the given conic at the common points of the ranges.*

Let $(ABC\ldots)$ and $(A'B'C'\ldots)$ be the two homographic ranges, and E, F their common points. Project the conic into a circle and the homographic axis EF to infinity. Then E, F are projected into the circular points.

Now in the second figure, AB' and $A'B$ meet on the homographic axis. Hence AB' and $A'B$ are parallel. So AC' and $A'C$ are parallel, and so on. Hence the arcs AA', BB', CC', ... are all equal. Hence the envelope of AA' is a concentric circle, i.e. a circle having double contact with the circle which is the projection of the given conic, at the circular points E, F. Hence in the original figure the envelope of AA' is a conic having double contact with the given conic at the common points of the two homographic ranges.

Ex. 1. *Two conics have double contact, and a tangent to one conic meets the other conic in A and A'. Show that A and A'*

generate homographic ranges, and find the common points of these ranges.

Ex. 2. *If* $(ABC\ldots)$ *and* $(A'B'C'\ldots)$ *be two homographic ranges on a conic, show that the locus of the poles of* AA', BB', ... *is a conic having double contact with the given conic.*

Ex. 3. *Show that the tangents at* $ABC\ldots$ *and* $A'B'C'\ldots$ *cut the homographic axis in homographic ranges.*

Ex. 4. *If* O *be the pole of the homographic axis of the two homographic ranges on a conic, then*

$$O(ABC\ldots) = O(A'B'C'\ldots).$$

Ex. 5. *If all but one of the vertices of a polygon move on fixed lines and all the sides touch a conic, the locus of the remaining vertex is a conic having double contact with the given conic.*

Reciprocate.

Ex. 6. *Two sides of a triangle inscribed in a conic pass through fixed points; show that the envelope of the third is a conic touching the given conic at the meets of the given conic with the join of the given points.*

Ex. 7. *A triangle* PQR *is inscribed in a conic;* PQ, PR *are in given directions; show that* QR *envelopes a conic.*

Ex. 8. *Inscribe in a given conic a polygon of any given number of sides, each side of which shall touch some fixed conic having double contact with the given conic.*

Ex. 9. *A conic is drawn through the common points* E, F *of two homographic ranges* A, B, C, ... *and* A', B', C', ... *on the same line. A pair of tangents moves so as to pass through a pair of points of these ranges. Show that the points of contact generate homographic ranges on the conic, whose common points are* E *and* F.

Ex. 10. *Also if, in Ex.* 9, P *be a variable point, and* PA *cut the conic in* a, p, *and* $A'p$ *cut the conic in* a'; *show that* aa' *envelopes a conic.*

Ex. 11. *If two conics* c_1 *and* c_2 *have double contact at the points* L *and* M, *and through* LM *be described any conic* c_3, *then the opposite two common chords of* c_1, c_3 *and* c_2, c_3 *meet on* LM.

Ex. 12. *If tangents at the two points* P, Q *on one of two conics, having double contact at* L *and* M, *meet the other in* AB *and* CD, *show that two of the chords* AC, AD, BC, BD *meet* PQ *on* LM, *and the other two meet* PQ *in points* UV *such that*

a conic can be drawn touching these chords at U and V and touching the conics at L and M.

Ex. 13. *If a tangent to a conic meet a homothetic and concentric conic in P and P', show that CP and CP' generate homographic pencils whose common lines are the common asymptotes, C being the common centre.*

XXIX

1. Project a given conic into a rectangular hyperbola, and a given point into a focus.

2. $A, B, C, D, A', B', C', D'$ are eight points on a conic. $AB, CD, A'B', C'D'$ are concurrent, and so are $BC, DA, B'C', D'A'$. Show that $CA, DB, C'A', D'B'$ meet in a point, and that a conic can be drawn touching $A'A, B'B, C'C, D'D$ at A, B, C, D.

3. Two triangles in perspective are circumscribed to a conic. Show that any transversal through the centre of perspective cuts the sides in pairs of points in involution.

4. A is a fixed point; P is any point on its polar for a given conic. The tangents from P meet a given line at Q and R. Show that AR and PQ meet on a fixed line.

5. A, B, C, D are four points on a conic. Show that the harmonic triangle of the quadrilateral AB, BC, CD, DA is generally not self-conjugate.

6. Points P, Q, R are taken on BC, CA, AB, and conics are described about $AQRLM$, $BRPLM$, $CPQLM$ where L, M are any two points. Show that these conics meet again in a third point.

7. The harmonic envelope of two parabolas whose axes are parallel is a parabola with axis parallel to these axes.

8. The harmonic locus of two equal circles which touch is two lines.

9. If the tangents at a common point of two conics are harmonic with two common chords, the harmonic envelope of the conics is two points.

10. A, B, C, D are four fixed points on a fixed conic. BC, DA meet at F, and AB, CD meet at G. A variable conic through A, C, F, G cuts the fixed conic again at P, Q.

Show that PQ passes through the pole of BD for the fixed conic.

11. Of two circles, the poles of the radical axis and the centres of similitude form a harmonic range.

12. Two conics c_1 and c_2 meet at B, C and touch at A. DEG touches c_1 at E and c_2 at G. DFH touches c_1 at F and c_2 at H. The tangent at A meets BC at K. Show that $A(KD, BC) = D(AK, EF) = K(FH, AC)$.

13. The locus of the point where the intercept of a variable tangent of a central conic between two fixed tangents is divided in a given ratio is a hyperbola whose asymptotes are parallel to the fixed tangents.

14. The point V on a conic is connected with two fixed points L and M. Show that chords of the conic which are divided harmonically by VL and VM pass through a fixed point O. Also as V varies, the locus of O is a conic touching the given conic at two points on the connector of the fixed points L and M.

15. The tangents at $A, B, C, \ldots$ to a conic meet a conic having double contact with the given conic at $P, Q, R, \ldots$ and $P', Q', R', \ldots$. Show that $(ABC\ldots) = (PQR\ldots)$.

16. A and B are fixed points on a conic and P and Q are variable points on this conic such that (AB, PQ) is constant. Show that AP and BQ intersect on a fixed conic having double contact with the given conic at A and B.

17. The triangle PQR is inscribed in a given conic. If the directions of PQ and PR are given, show that QR envelopes a conic having the same asymptotes as the given conic.

18. A chord through the fixed point O meets a conic at P, P' and on it is taken a point Q such that (OQ, PP') is constant. If PP' meet the locus of Q again at Q', show that $(OQ, PP') \times (OQ', PP') = 1$.

19. Four points $PQRS$ are taken on a conic. PR and QS meet at L. Any point M is taken and PM, QM meet the conic at T, U. Prove that ST and RU meet on LM. Prove also that, if PS and QR meet on LM, then PQ, SR, UT are concurrent.

20. Any conic drawn through a point and the points of

contact of tangents from the point to a given conic is such that quadrangles can be inscribed in it which are circumscribed to the given conic.

21. AA', BB', CC' are the opposite vertices of a quadrilateral circumscribed to a conic. O is any point. OA cuts the polar of A at a, OB cuts the polar of B at b, and so on. Show that the seven points O, a, a', b, b', c, c' lie on a conic.

22. A, B, C, D are four points on a conic, and AC, BD intersect at O. Prove that, if any line through O meets CD at P and BC at Q, and AP, AQ meet the conic at E, F respectively, then BE, FD, OP and the tangent at C are concurrent.

23. Points A, B, C are such that an infinite number of triangles can be inscribed in a given conic whose respective sides shall pass through these points. Show that ABC is a self-conjugate triangle for the conic.

24. A system of conics passes through the points A, B, C, D. Through A and B are drawn the lines l and m. Show that the conics determine on l and m ranges in perspective. Also state the corresponding theorem when both l and m are drawn through A.

* CHAPTER XXX *

GENERALIZATION BY PROJECTION

1. IN the previous chapter we have investigated theorems by projecting the given figure into the simplest possible figure. In this chapter we shall deal with the converse process, viz. of deriving from a given theorem the most general theorem which can be deduced by a projection, real and imaginary. This process is called *Generalizing by Projection.*

In our present advanced state of knowledge of Pure Geometry, Generalization by Projection is not a very valuable instrument of research. In fact the student will often find that it is more easy to prove the generalized theorem than the given theorem.

Many things are as general already as they can be. For instance, if we generalize by projection a point, a line, a conic, a harmonic range, a range having a given cross ratio, two conics having double contact, and so on, we obtain the same thing.

2. The properties of any figure have an intimate relation with the circular points ∞, ∞'. Hence the generalized figure will have an intimate relation with the projections of the circular points. But in the second figure there will also be a pair of circular points. Hence, to avoid confusion, we shall call the projections of the circular points ϖ and ϖ'.

3. Since any two points can be projected into the circular points, *the circular points* generalize into any two points ϖ and ϖ', real or imaginary.

Since a pair of circular lines pass through the circular

points, *a pair of circular lines* generalizes into a pair of lines, one through ϖ and one through ϖ'.

Since all circles pass through the circular points, *a circle* generalizes into a conic which passes through ϖ and ϖ', where ϖ and ϖ' are any two points.

Since concentric circles touch one another at the circular points, *concentric circles* generalize into conics touching one another at ϖ and at ϖ'.

Since the line at infinity touches a parabola, *a parabola* generalizes into a conic touching the line $\varpi\varpi'$.

Notice that we cannot generalize the distinction between *a hyperbola* and *an ellipse*; for by an imaginary projection a pair of real points may be projected into a pair of imaginary points and vice versa.

Since a rectangular hyperbola is a conic for which the circular points are conjugate, *a rectangular hyperbola* generalizes into a conic for which ϖ, ϖ' are a pair of conjugate points.

Since the centre of a conic is the pole of the line at infinity, *the centre of a conic* generalizes into the pole of the line $\varpi\varpi'$.

Hence *a circle on AB as diameter* generalizes into a conic passing through $AB\varpi\varpi'$, and such that the pole of the line $\varpi\varpi'$ is on AB.

Since parallel lines meet on the line at infinity, *parallel lines* generalize into lines which meet at a point on the line $\varpi\varpi'$.

Note that throughout this chapter, ϖ and ϖ' are any two points, real or imaginary.

4. If B bisects the segment AC, then the range $(AC, B\Omega)$ is harmonic; hence '*B bisects AC*' generalizes into 'If AC meet $\varpi\varpi'$ in I, then B is such that (AC, BI) is harmonic, ϖ and ϖ' being any two points.'

Generalize by Projection the theorem—'Given two concentric circles, any chord of one which touches the other is bisected at the point of contact.'

The result is—'Given two conics touching one another at any two points ϖ and ϖ', if any chord PP' of one touch the other at Q and meet $\varpi\varpi'$ in I, then (PP', QI) is harmonic.'

Or, without mentioning ϖ and ϖ',—'Given two conics having double contact, if any chord PP' of one touch the other at Q and meet the chord of contact in I, then (PP', QI) is harmonic.'

The student should convince himself by trial that the second theorem can be projected into the first, and that the second theorem is the most general theorem which can be projected into the first.

Generalize by Projection the following theorems—

Ex. 1. *The middle points of parallel chords of a circle lie on a line which passes through the centre of the circle.*

Ex. 2. *If the directions of two sides of a triangle inscribed in a circle are given, then the envelope of the third is a concentric circle.*

5. If AVA' is a right angle, then VA and VA' divide the segment joining the circular points harmonically; hence *a right angle AVA'* generalizes into an angle AVA', such that VA and VA' divide the segment joining any two points ϖ, ϖ' harmonically.

Generalize by Projection the theorem—'The perpendiculars to the sides of a triangle at the middle points of the sides meet at the centre of the circumcircle.'

The result is—'If the sides BC, CA, AB of a triangle meet the segment joining any two points ϖ and ϖ' in L, M, N; and if X, Y, Z be taken such that $(\varpi\varpi', XL)$, $(\varpi\varpi', YM)$, $(\varpi\varpi', ZN)$ are harmonic; and if D, E, F be taken such that (BC, DL), (CA, EM), (AB, FN) are harmonic; then DX, EY, FZ meet at the pole of $\varpi\varpi'$ for the conic which passes through $ABC\varpi\varpi'$.'

Generalize by Projection the following theorems—

Ex. 1. *A tangent of a circle is perpendicular to the radius to the point of contact.*

Ex. 2. *The feet of the perpendiculars from any point on a circle on the sides of an inscribed triangle are collinear.*

Ex. 3. *The locus of the meet of perpendicular tangents of a conic is a concentric circle.*

Ex. 4. *The circle about any triangle self-conjugate for a conic is orthogonal to its director circle.*

Ex. 5. *The chords of a conic which subtend a right angle at a fixed point on the conic pass through a fixed point on the normal at the point.*

Ex. 6. *If a triangle PQR, right-angled at P, be inscribed in a rectangular hyperbola, the tangent at P is the perpendicular from P on QR.*

6. Since all circles pass through the circular points, *a system of circles* generalizes into a system of conics passing through the same two points (ϖ and ϖ').

Since coaxal circles pass through the same four points of which two are the circular points, *coaxal circles* generalize into a system of conics which pass through the same four points (of which two are ϖ and ϖ').

Since the limiting points of a system of coaxal circles are the two vertices of the common self-conjugate triangle which lie on the line joining the poles of $\infty\infty'$, *the limiting points* generalize into the two vertices of the common self-conjugate triangle of a system of four-point conics which lie on the line joining the poles of any common chord ($\varpi\varpi'$), i. e. they generalize into any two vertices of the common self-conjugate triangle.

Since the centres of similitude of two circles are the two intersections of common tangents which lie on the line joining the poles of $\infty\infty'$ for the circles, *the centres of similitude of two circles* generalize into the two intersections of common tangents of two conics (through ϖ and ϖ') which lie on the line joining the poles of any common chord ($\varpi\varpi'$) for the conics, i. e. they generalize into any pair of opposite common apexes of two conics.

7. *Generalize by Projection the theorem—'Any common tangent of two circles subtends a right angle at either limiting point.'*

The result is—'If ϖ and ϖ' be any two common points of two conics, and if L and L' be the two vertices of the common self-conjugate triangle which are collinear with the poles of ϖ, ϖ', then any common tangent of the conics subtends at L (and at L') an angle whose rays divide the segment $\varpi\varpi'$ harmonically.'

In other words,—'Any common tangent of two conics subtends at any vertex of the common self-conjugate triangle an angle which divides harmonically every common chord which does not pass through this vertex.'

Generalize by Projection the theorems—

Ex. 1. *Any transversal meets a system of coaxal circles in pairs of points in involution.*

Ex. 2. *The circle of similitude of two circles is coaxal with them.*

8. Since a focus of a conic is one of the four meets of the tangents from the circular points to the conic, *a focus of a conic* generalizes into one of the meets of the tangents from any two points (ϖ and ϖ') to the conic.

The two foci of a conic generalize into a pair of opposite intersections of the tangents from any two points (ϖ and ϖ'); and the *axes* generalize into the lines joining these opposite pairs of intersections, and these generalized axes cut the conic in the generalized *vertices*.

Since the line joining the circular points touches a parabola, *the focus of a parabola* generalizes into the meet of tangents from any two points (ϖ and ϖ') lying on any tangent of a conic. If $\varpi\varpi'$ touches the generalized parabola at I, then SI, the generalized *axis*, cuts the conic again at A, the generalized *vertex* of the parabola.

Since confocal conics touch the same four tangents from the circular points (viz. $S\infty$, $S'\infty$, $S\infty'$, $S'\infty'$), *confocal conics* generalize into conics inscribed in the same quadrilateral (of which ϖ and ϖ' are a pair of opposite vertices).

Since conics which have the same focus S and the same corresponding directrix l touch $S\infty$, $S\infty'$, where l

meets these lines, *conics which have the same focus S and the same corresponding directrix l* generalize into conics having double contact, the common tangents passing through S (and through ϖ and ϖ'), and touching the conics at points on l.

A conic having S as focus generalizes into a conic touching any two lines ($S\varpi$ and $S\varpi'$) through S.

9. *Generalize by Projection the theorem—'The circle which circumscribes a triangle whose sides touch a parabola passes through the focus of the parabola.'*

The result is—'The conic which passes through the points A, B, C, ϖ, ϖ', where ϖ and ϖ' are any two points, and A, B, C are the vertices of a triangle whose sides touch a conic which touches the line $\varpi\varpi'$, passes through the meet of tangents to the latter conic from ϖ and ϖ'.'

In other words—'The conic, which passes through five out of the six vertices of two triangles which circumscribe a given conic, passes through the sixth also.'

Generalize by Projection the following theorems—

Ex. 1. *Any line through a focus of a conic is perpendicular to the line joining its pole to the focus.*

Ex. 2. *Given a focus and two tangents of a conic, the locus of the other focus is a line.*

Ex. 3. *The locus of the centre of a circle which touches two given circles is a conic having the centres of the circles as foci.*

Ex. 4. *The locus of the centre of a circle which passes through a fixed point and touches a fixed line is a parabola of which the point is the focus.*

Ex. 5. *Confocal conics cut at right angles.*

Ex. 6. *The envelope of the polar of a given point for a system of confocals is a parabola touching the axes of the confocals and having the given point on its directrix.*

10. Since the rays of an angle of given size divide the segment joining the circular points in a given cross ratio, *a constant angle* generalizes into an angle whose rays divide

the segment joining any two points (ϖ and ϖ') in a constant cross ratio.

Since *similar conics* have the same angle between the asymptotes, they generalize into conics which cut $\varpi\varpi'$ in a constant cross ratio; for a conic and its asymptotes cut the line at infinity in the same points.

Generalize by Projection the theorem—'The envelope of a chord of a conic which subtends a constant angle at a focus S is another conic having S as focus; and the two conics have the same directrix corresponding to S.'

The result is—'The envelope of a chord of a conic which subtends at S, one of the meets of a tangent from any point ϖ with a tangent from any point ϖ', an angle whose rays divide $\varpi\varpi'$ in a constant cross ratio, is another conic, touching $S\varpi$ and $S\varpi'$; and the two conics have the same polar of S.'

In other words—'If SQ and SR be the tangents from any point S to a conic, the envelope of a chord PP' of the conic such that $S(QRPP')$ is constant, is a conic having double contact with the given conic at the points of contact of SQ and SR.'

Ex. 1. *Generalize—'a regular polygon.'*

A regular polygon may be defined as a polygon which can be inscribed in a circle so that each side subtends the same angle at the centre of the circle.

Generalize by Projection the following theorems—

Ex. 2. *The envelope of a chord of a circle which subtends a given angle at any point of the circle is a concentric circle.*

Ex. 3. *If from a fixed point O, OP be drawn to a given circle, and TP be drawn making the angle TPO constant, the envelope of TP is a conic with O as focus.*

Ex. 4. *If from a focus of a conic a line be drawn making a given angle with a tangent, the locus of the point of intersection is a circle.*

Ex. 5. *The locus of the intersection of tangents to a parabola which meet at a given angle is a hyperbola having the same focus and corresponding directrix.*

11. *Generalize—'The bisectors of an angle.'*

If AD, AE are the bisectors of the angle BAC, then $A\,(BC,\ DE)$ is harmonic, and also $A\,(\infty\infty',\ DE)$ since EAD is a right angle. Hence the bisectors of the angle BAC generalize into the double lines of the involution

$$A\,(BC,\ \varpi\varpi'),$$

where ϖ and ϖ' are any two points.

Ex. *Generalize by Projection—'The pairs of tangents from any point to a system of confocals have the same bisectors.'*

12. *Generalize—'a segment divided in a given ratio.'*

Let AB be divided at C in a given ratio. Then $AC : CB$ is constant; hence $(AB,\ C\Omega)$ is constant, where Ω is the point at infinity upon AB. Hence *a segment AB divided in a given ratio at C* generalizes into a segment AB divided at C so that $(AB,\ CI)$ is constant, I being the meet of AB and the segment joining any two points (ϖ and ϖ').

Ex. 1. *Generalize the equation $AB + BC + CA = 0$ connecting three collinear points.*

The given equation may be written

$$-(AC,\ B\Omega) + 1 - (AB,\ C\Omega) = 0.$$

This generalizes into $-(AC,\ BI) + 1 - (AB,\ CI) = 0$,

i. e. into $AB\,.\,CI + AI\,.\,BC + AC\,.\,IB = 0.$

Hence the generalized theorem is—'If A, B, C, D be any four collinear points, then

$$AB\,.\,CD + AC\,.\,DB + AD\,.\,BC = 0.\text{'}$$

Ex. 2. *If $ABCD$ be collinear, show that the ratio $AB \div CD$ generalizes into* $-(BC,\ AE) \div (DA,\ CE)$.

13. *Two fixed points A and B on a conic are joined to a variable point P on the conic, and the intercept QR cut off from a given line l by PA and PB is divided at M in a given ratio; show that the envelope of PM is a conic touching parallels to l through A and B.*

Let Ω be the point at infinity on l. Then $(QR,\ M\Omega)$ is a given cross ratio. Hence $P\,(AB,\ M\Omega)$ is given. Project A and B into the circular points and let I be the projection

of Ω. Then $P(\infty\infty', MI)$ is given, i. e. IPM is a given angle.

Hence the theorem becomes—'A fixed point I is joined to a variable point P on a circle, and PM is drawn making a given angle with IP: show that the envelope of PM is a conic touching $I\infty$ and $I\infty'$, i. e. is a conic having I as focus.' And this is true (see VIII. 17). Hence the original theorem is true.

Generalization by Reciprocation.

14. If we first generalize a given theorem by projection and then reciprocate the generalized theorem, we obtain another general theorem. This process is called *Generalizing by Projection and Reciprocation*, or briefly *Generalizing by Reciprocation.*

Generalize by Reciprocation the theorem—'All normals to a circle pass through the centre of the circle.'

Generalizing by Projection we get—'If t be the tangent at any point P of a conic which passes through any two points ϖ, ϖ', and if the line n be taken such that t and n are harmonic with $P\varpi$ and $P\varpi'$, then n passes through the pole of $\varpi\varpi'$ for the conic.'

Reciprocating this theorem we get—'If on the tangent at any point T of a conic, a point N be taken such that the segment TN is divided harmonically by the tangents from the fixed point O, then N lies on the polar of O for the conic.' This is the required theorem.

Ex. *Generalize by Projection and Reciprocation the theorem—'The envelope of a chord of a circle which subtends a constant angle at the centre is a concentric circle.'*

* CHAPTER XXXI *

HOMOLOGY

1. Two *figures* in the same plane are said to be *in homology* which possess the following properties. To every point in one figure corresponds a point in the other figure, and to every line in one figure corresponds a line in the other figure. Every two corresponding points are collinear with a fixed point called *the centre of homology*, and every two corresponding lines are concurrent with a fixed line called *the axis of homology*. The line joining any two points of one figure corresponds to the line joining the two corresponding points of the other figure. The point of intersection of any two lines of one figure corresponds to the point of intersection of the two corresponding lines of the other figure.

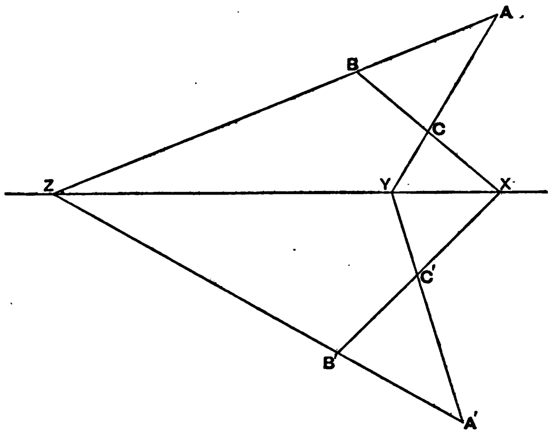
The two figures are said to be homological, and each is called the *homologue* of the other. The figures may also be said to be in plane perspective; and the centre of homology is then called the centre of perspective, and the axis of homology is called the axis of perspective.

2. Homological figures exist, for—

If we take two figures in different planes, each of which is the projection of the other, and if we rotate one of the figures about the meet of the two planes until the planes coincide, then the figures will be homological.

For let ABC be the projections of $A'B'C'$ from the vertex V. Then AA', BB', CC' meet in V. That is, the triangles ABC, $A'B'C'$ (in different planes) are copolar. Hence they are coaxal; i.e. BC, $B'C'$ meet in X, and CA, $C'A'$ meet in Y, and AB, $A'B'$ meet in Z on the meet of the two planes. Similarly every two lines which are the projections, each of the other, meet on the intersection of the two planes.

Now rotate one figure about the line XYZ until the two figures are in the same plane. Then the two triangles are still coaxal (for BC, $B'C'$ still meet at X, and so for the rest). Hence the two triangles are also copolar; i.e. AA', BB', CC' meet in a point. Call this point O. Then O may be defined as the meet of AA' and BB', and we have proved that every other line such as CC' passes through O.



Now the two figures are in the same plane. Also to every point in one figure corresponds a point in the other figure, viz. the point which was its projection; and to every line corresponds a line, viz. its former projection. Also, corresponding points are concurrent with a fixed point O, and corresponding lines are collinear with a fixed line XYZ. Also, the join of two points corresponds to the join of the corresponding points; for in the former figure the one is the projection of the other. For the same reason, the meet of two lines corresponds to the meet of the corresponding lines.

Hence the two figures are homological.

Notice that in this way we get two sets of homological figures, the angle of rotation differing by two right angles.

The reader should follow all this on the model at the end of the book.

3. *If two figures are homological, and we turn one of them about the axis of homology, the figures will be the projections, each of the other.*

For suppose the three lines BC, CA, AB in one figure to be homologous to $B'C'$, $C'A'$, $A'B'$ in the other figure. Let BC, $B'C'$ meet in X, let CA, $C'A'$ meet in Y, and let AB, $A'B'$ meet in Z. Rotate one of the figures about the axis of homology XYZ, so that the figures may be in different planes. The figures will now be each the projection of the other.

For the triangles ABC, $A'B'C'$ (in different planes) are coaxal; hence they are copolar. Hence AA', BB', CC' meet in a point V. This point V may be defined as the meet of AA' and BB'; and we have proved that in the displaced position the join CC' of any two homologous points passes through a fixed point V. Hence the homological figures in the displaced position are projections, each of the other.

A homologue of a conic is a conic.

For after rotating one figure about the axis of homology, the figures are each the projection of the other; and the projection of a conic is a conic.

A homologue of a figure has all the properties of a projection of the figure.

For it can be placed so as to be a projection of the figure.

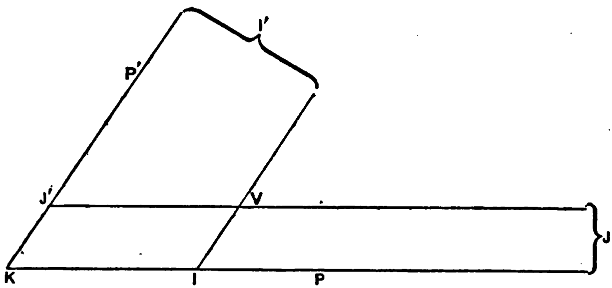
Hence *a range and the homologous range are homographic; also a pencil and the homologous pencil are homographic.*

4. *If one of two figures in perspective (i. e. either homological or each the projection of the other), be rotated about the axis of perspective, the figures will be in perspective in every position; and the locus of the centre of perspective is a circle.*

For take any two corresponding triangles ABC and $A'B'C'$. Then in every position these triangles will remain coaxal; hence in any position they will be copolar, i.e. CC' will pass through the fixed point V determined as the meet of AA' and BB'. Hence the figures will be in perspective in any position obtained by rotating one figure about the axis of perspective.

To find the locus of V, take any position of V, and through V draw a plane $P'KP$ at right angles to the planes of the figures, cutting them in KP' and KP.

Let a parallel to KP' through V cut KP in I, and a parallel to KP through V cut KP' in J'. Let the point at infinity on KP be called J, and the point at infinity on KP' be called I'.



Then, since $J'V$ and KI are parallel, we see that $J'V$ passes through J, i.e. J' is the projection of J for this position of V; and so I is the projection of I'.

Now rotate the moving plane about the axis of perspective into any other position. The new position of the centre of perspective (or vertex of projection) is got by joining any two pairs AA', BB' of corresponding points. Hence in the new position II' and JJ' will cut in V. Also KI is still parallel to $J'V$, for J is at infinity; so IV is parallel to KJ'. Also if KI is the trace on the fixed plane, then KI is constant in magnitude and position. Also KJ' is constant in magnitude, although it changes its position by rotation about the axis of perspective. It follows that $KIVJ'$ is a parallelogram, in which I is fixed, and IV is given in magnitude. Hence the locus of V is a circle in a plane perpendicular to the planes of the figures, with centre I and radius KJ'.

To form a clear conception of figures in homology, imagine that they are the projections, each of the other, the vertex of projection very nearly coinciding with the centre

of homology, and the planes of the figures very nearly coinciding with one another.

Since figures in homology can be obtained from figures in projection by turning one of the planes about the axis of projection until the planes coincide, it follows that figures in homology have two *equiangular points* and two *equisegmental* lines. Suppose, in the figure of IV. 8, KF' rotates about K until F' coincides with F. Then since in the original figure OI is parallel to KF', we have $OI:IF::KF':KF$; hence $IO = IF$. Hence in the new figure O also coincides with F. Hence (as we should expect) one pair of the equiangular points coincide at the centre of homology. Also $IE = IO$ and $OJ' = J'E'$; which show that, in the figures in homology, the other equiangular points are the reflexions of the centre of homology in the vanishing lines.

Again, in figures in projection, the equisegmental lines are the reflexions of the axis of projection in the vanishing lines. Hence, in figures in homology also, the equisegmental lines are the reflexions of the axis of perspective in the vanishing lines. Of course the axis of perspective is itself an equisegmental line.

Notice that in each case we reflect, in the vanishing lines, the point or line which obviously possesses the property under consideration.

5. *Coaxal figures are copolar, and copolar figures are coaxal; that is to say, if two figures, (in the same plane or not,) correspond, point to point, line to line, meet of two lines to meet of corresponding lines, and join of two points to join of corresponding points; then, if corresponding lines cut on a fixed line, the joins of corresponding points will pass through a fixed point, and if the joins of corresponding points pass through a fixed point, corresponding lines will cut on a fixed line.*

Coaxal figures are copolar. Take two fixed points A, B in one figure, and let A', B' be the corresponding points in the other figure. Take any variable point P in one figure, and

let P' be the corresponding point in the other figure. Then, by definition, AP, $A'P'$ are corresponding lines, for they join corresponding points; hence AP and $A'P'$ meet on the axis. Similarly BP, $B'P'$ meet on the axis; and AB, $A'B'$ meet on the axis. Hence the triangles ABP, $A'B'P'$ are coaxal, and therefore copolar. Hence AA', BB', PP' meet in a point, i.e. PP' passes through a fixed point, viz. the meet of AA' and BB'.

Copolar figures are coaxal. Take two fixed lines, viz. AP and AQ, and a variable line PQ in one figure, and let $A'P'$, $A'Q'$, $P'Q'$ be the corresponding lines in the other figure. Then the points A, P, Q correspond to A', P', Q'. Hence the triangles APQ, $A'P'Q'$ are copolar, and therefore coaxal. Hence PQ, $P'Q'$ meet on a fixed line, viz. the join of the meets of AP, $A'P'$ and of AQ, $A'Q'$. Hence the figures are coaxal.

Ex. 1. *If one of two figures in homology be turned through two right angles about an axis which passes through the centre of homology and is perpendicular to the plane of the figures, the figures will again be in homology.*

For the figures will be again copolar.

Ex. 2. *Given two homological figures $ABC\ldots$, $A'B'C'\ldots$, let $A''B''C''\ldots$ be a projection of $ABC\ldots$ on any plane through the axis of homology; then will $A''B''C''\ldots$ be also a projection of $A'B'C'\ldots$, and the vertices of projection and the centre of homology will be collinear.*

For $(A''\ldots)$ and $(A'\ldots)$ are coaxal, and hence copolar. Let AA' and BB' meet at O, AA'' and BB'' at V, and $A'A''$ and $B'B''$ at V'. Then, when A is at O, A' is at O. Hence VO is one position of $A'A''$, and hence passes through V'.

This construction enables us to place any two homological figures in projection with the same figure.

Ex. 3. *Show that the two complete quadrangles determined by the points $ABCD$ and $A'B'C'D'$ will be homological provided the five points of intersection of AB with $A'B'$, of BC with $B'C'$, of CA with $C'A'$, of AD with $A'D'$, and of BD with $B'D'$ are collinear.*

Ex. 4. *Show that the two complete quadrilaterals whose vertices are $ABCDEF$ and $A'B'C'D'E'F'$ will be homological if AA', BB', CC', DD', EE' meet in a point.*

Ex. 5. *The sides PQ, QR, RP of a variable triangle pass through fixed points C, A, B in a line. Q moves on a fixed line. Show that P and R describe homological curves.*

For PR and $P'R'$ pass through the fixed point B, and RR', PP' meet on the fixed line QQ', $P'Q'R'$ being a second position of PQR.

Ex. 6. *If the axis of homology be at infinity, show* (i) *that corresponding lines are parallel,* (ii) *that corresponding sides of the figures are proportional,* (iii) *that corresponding angles of the figures are equal.*

Such figures are called *homothetic figures*, and the centre of homology in this case is called *the centre of similitude*, and the constant ratio of corresponding sides is called *the ratio of similitude.*

Ex. 7. *If, with any vertex of projection, we project homological figures on to any plane, we obtain homological figures; and if the plane of projection be taken parallel to the plane containing the vertex of projection and the axis of homology, we obtain homothetic figures.*

Hence homological figures might have been defined as the projections of homothetic figures.

Ex. 8. *If the centre of homology be at infinity, show that the joins of corresponding points are all parallel; and that if one figure be rotated about the axis of homology, the vertex of projection will always be at infinity.*

This may be called *parallel homology.*

Ex. 9. *In parallel homology, show that to a point at infinity corresponds a point at infinity, and that the line at infinity corresponds to itself.*

Ex. 10. *In parallel homology, show that a parallelogram corresponds to a parallelogram.*

Ex. 11. *In parallel homology, show that, when rotated about the axis of homology into different planes, the figures have the same orthogonal projection; and that the ratios of two areas is the same as that of the corresponding areas.*

6. The abbreviation c. of h. will be used for centre of homology, and a. of h. for axis of homology.

Given the c. of h. and the a. of h. and a pair of corresponding points, construct the homologue of a given point.

Let O be the c. of h., and let A' be the given homologue

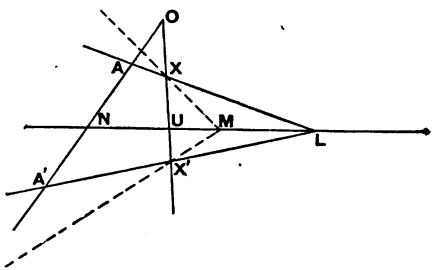
of A. To find the homologue of X; let AX cut the a. of h. in L, then LA' cuts OX in the required point X'.

With the same data, construct the homologue of a given line.

Draw through O any transversal cutting the given line in X; construct the homologue X' of X, then the join of X' to the point M, where the given line cuts the a. of h., is the homologue of the given line.

Given the c. of h. and the a. of h. and a pair of corresponding lines, construct the homologue (i) *of a given point,* (ii) *of a given line.*

Let any transversal through O cut the given lines in A



and A'. Then A, A' are corresponding points, and we may proceed as above.

Given the c. of h. and the a. of h. and a pair of corresponding points, one of which is at infinity, construct the homologue of a given point.

LX' is parallel to AA', if A' is at infinity.

Ex. *Given the homologues A', B', C' of three points A, B, C; construct the homologue of a given point D.*

The triangles give the centre and axis of homology.

7. *The homologue of the c. of h. is the c. of h.; the homologue of any point on the a. of h. is the point itself; if the homologue of any other point be itself, then the homologue of every point is itself.*

For let us construct the homologue of O. We draw AO cutting the a. of h. in N; we draw NA' cutting OO in the required point. Now OO is indeterminate, but NA' cuts

every line through O in O, and hence cuts OO in O. Hence the homologue of O is O.

Next, let us construct the homologue of any point L on the a. of h. We draw AL cutting the a. of h. in L; we draw $A'L$ cutting OL in the required point. Hence the homologue of L is L.

Lastly, suppose a point (which is not at the c. of h. nor on the a. of h.) to coincide with its homologue. Take these as the points A, A' in the above construction. To construct the homologue of X, we draw AX cutting the a. of h. in L; then $A'L$ cuts OX in the required point X'. Hence X' coincides with X, for $A'L$ coincides with AL, i.e. with AX.

The homologue of the a. of h. is the a. of h.; the homologue of every line through the c. of h. is the line itself; if the homologue of any other line is the line itself the homologue of every line is the line itself.

We have shown that the homologue of every point on the a. of h. is the point itself; hence the homologue of the a. of h. is the a. of h.

Again, if OA be any line through A, the homologue of OA is the line OA' where A' is the homologue of A; i.e. is OA itself.

Again, if l corresponds to itself, any line through O will cut l in a point A which corresponds to itself; hence by the first part, every point and therefore every line corresponds to itself.

8. The homologue of a point at infinity of one figure is called *a vanishing point* of the homological figure.

The homologue of the line at infinity considered as belonging to one of the figures is called *the vanishing line* of the homological figure.

All the vanishing points of either figure lie on the vanishing line of that figure.

For a vanishing point is the homologue of a point on the line at infinity of the other figure, and hence lies on the homologue of the line at infinity.

Each vanishing line is parallel to the a. of h.

For corresponding lines meet on the a. of h. Hence a vanishing line meets the a. of h. at a point on the line at infinity, i.e. a vanishing line is parallel to the a. of h.

Ex. 1. *If any transversal through O cut the axis in N, and the vanishing lines in I and J', then $OI = J'N$.*

For $(NI, O\Omega) = (N\Omega', OJ')$.

Ex. 2. *The product of the perpendiculars from any two homologous points, each to the vanishing line of its figure, is constant.*

Let PQ cut its vanishing line at I, and $P'Q'$ cut its vanishing line at J'; then $(PQ, I\Omega) = (P'Q', \Omega'J')$.

Ex. 3. *Given a parallelogram $ABCD$, prove the following construction for drawing through a given point E a parallel to a given line l—Let AB, CD, AC, BC, AD cut l in K, L, M, N, R. Through M draw any line cutting EK, EL in A', C'. Let RA' and NC' cut in F. Then EF is parallel to l.*

For EF is the vanishing line.

9. *Given the c. of h., the a. of h., and a pair of corresponding points; construct the vanishing lines.*

Let AA' be the pair of corresponding points. Let us first construct the homologue of the line at infinity, considered to belong to the same figure as A. In the construction of § 6, X and M are both at infinity. Hence the construction is—Through the c. of h. O, draw any line OX (X being at infinity). Through A draw AL parallel to OX, cutting the a. of h. in L; then LA' cuts OX in X'; and the required line is $X'M$, i.e. a parallel through X' to the a. of h.

Similarly we construct the vanishing line of the other figure.

Given the c. of h., the a. of h., and one vanishing line, to construct the homologue of a given point.

Let any transversal through the c. of h. cut the vanishing line in A. Then the homologue of the point A is the point at infinity A' on OA.

Two cases arise. (i) The given point X belongs to the same figure as the finite point A. Let AX cut the a. of h. in L; draw through L a parallel to OA to cut OX in X'.

Then X' is the homologue of X. (ii) The given point X' belongs to the same figure as the point at infinity A'. Through X' draw a parallel to OA cutting the a. of h. in L. Then AL cuts OX' in the required point X.

Ex. *Given the c. of h. and the a. of h. and one vanishing line, construct the other vanishing line.*

10. *The angle between two lines in one figure is equal to the angle subtended at the c. of h. by the vanishing points of the homologous lines.*

Let AP and AQ be the given lines, P and Q being at infinity. Then P' and Q' are the vanishing points of the homologous lines $A'P'$ and $A'Q'$. Also OP' is parallel to AP, and OQ' to AQ. Hence the angles $P'OQ'$ and PAQ are equal.

11. *Construct the homologue of a given conic, so that the homologue of a given point S shall be a focus.*

Take any line as a. of h., and any parallel line as vanishing line; and let two conjugate lines at S meet the vanishing line in P and Q, and let two other conjugate lines at S meet it in U and V. On PQ and UV as diameters describe circles, and take either of the intersections of these circles as c. of h.

Then since the vanishing points P and Q of the lines SP and SQ subtend a right angle at the c. of h., the homologues $S'P'$, $S'Q'$ will be at right angles. So $S'U'$, $S'V'$ will be at right angles. Hence at S' we shall have two pairs of conjugate lines at right angles. Hence S' is a focus of the homologous conic.

12. *The homologue of a conic, taking a focus as c. of h. and the corresponding directrix as vanishing line and any parallel as a. of h., is a circle, of which the focus is the centre.*

Let S be the given focus and XM the corresponding directrix. With S as c. of h. and XM as vanishing line, and any parallel line as a. of h., describe a homologue of the given conic. The homologue of S is S, and of XM is the

line at infinity; hence in the homologous conic, S is the pole of the line at infinity, i.e. S is the centre of the homologous conic.

Let SP, SP' be a pair of conjugate diameters of the homologous conic. The homologue of SP is SP, the homologue of SP' is SP'; and the homologues of conjugate lines are conjugate lines. Hence in the given figure, SP and SP' are conjugate lines; and S is the focus, hence SP and SP' are perpendicular. Hence every pair SP, SP' of conjugate diameters of the homologous conic is orthogonal. Hence the homologous conic is a circle. And we have already proved that the focus is the centre of the circle.

Note that the homologue of an angle at S is an equal (in fact, coincident) angle at S.

This case of homology is the limit of Focal Projection when the two figures are in the same plane.

Ex. 1. *Any homologue of a conic, taking a focus S as c. of h., is a conic with S as focus; and the homologue is a circle only if the vanishing line is the corresponding directrix.*

Ex. 2. *Any homologue of a conic, taking the polar of a given point P as vanishing line, is a conic with P as centre; and the homologue is a circle only if P is a focus of the given conic.*

13. *If two curves be in homology, the c. of h. must be a meet of common tangents, and the a. of h. must be a join of common points.*

For let OT be a tangent from the c. of h. O to one of the curves. Let OPQ be a chord of the curve very near OT. Then OPQ meets the homologous curve in the homologous points P', Q'. Now let P and Q coincide in T; then P' and Q' also coincide, in T' the homologue of T. Hence OT' touches the homologous curve.

Again, let L be one of the points where one of the curves cuts the a. of h. Then L, being on the a. of h., is its own homologue. Hence the homologous curve passes through L,

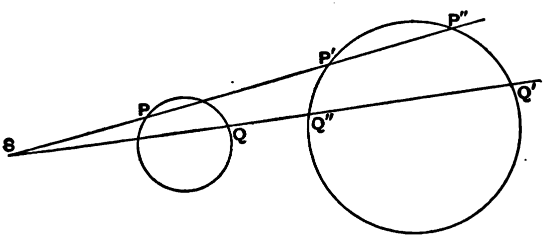
Hence if two curves are in homology, the c. of h. must be looked for among the meets of common tangents; and the a. of h. must be looked for among the joins of common points.

14. *Any two circles are homological in four real ways.*

Let S be either of the centres of similitude of the two circles. Take any point P on one circle, and let SP cut the other circle in P' and P''. Then one of these points, and only one (viz. P' in the figure), possesses the property that $SP:SP'$ is the ratio of the radii. We may call P, P' similar points, and P, P'' non-similar points.

If we take either centre of similitude as centre of homology and the straight line at infinity as axis of homology, then the circles are homological, each point being homologous to its similar point.

For take any two pairs of similar points, viz. P, P' and Q, Q'. Then $SP:SP'::SQ:SQ'$; hence PQ is parallel to $P'Q'$,



i.e. every chord joining two points on one circle is parallel to the chord joining the similar points on the other circle. Hence the two circles are homological, the straight line at infinity being the axis of homology, and similar points being homologous points.

If we take either centre of similitude as centre of homology and the radical axis as the axis of homology, then the circles are homological, each point being homologous to its non-similar point.

For take any two pairs of non-similar points, viz. P, P'' and Q, Q''. Then $SP:SP'::SQ:SQ'$, and

$$SP'.SP'' = SQ'.SQ''.$$

Hence $SP.SP'' = SQ.SQ''$. Hence $PP''QQ''$ are concyclic; hence, if PQ, $P''Q''$ meet in X, we have

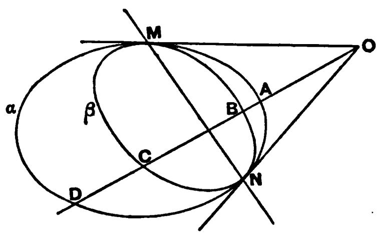
$$XP.XQ = XP''.XQ'',$$

i. e. X has the same power for both circles, i. e. X is on the radical axis of the circles. Hence we have proved that the chord joining any two points on one circle and the chord joining the non-similar points on the other circle meet on the radical axis of the circles, which is therefore the axis of homology.

Hence, since with either centre of similitude we may take the straight line at infinity or the radical axis, the circles are in homology in four real ways.

15. *Two conics which have double contact are homological in two ways, the c. of h. being the common pole and the a. of h. the common polar in both cases.*

Let O be the common pole and MN the common polar, the points M and N being on both conics. Let any line through O cut one conic a in A, D and the other conic β in B, C. Then a is determined by the five points $AMMNN$, the points MM being on OM and the points NN being on ON. Now form the homologue of a, taking O as c. of h., MN as a. of h., and B as the homologue of A. The homologue of a conic is a conic. The homologues of the points $AMMNN$ are the points $BMMNN$. Hence the homologue of a is the conic through $BMMNN$, i. e. is the conic β.



Again, with the same c. of h. and a. of h., but with C as the homologue of A, form the homologue of a. The homologue is now the conic through $CMMNN$, i. e. is the conic β.

Now in the first case C and D are homologous for they are collinear with O; so in the second case B and D are homologous. So there are two ways only in which the conics are homological.

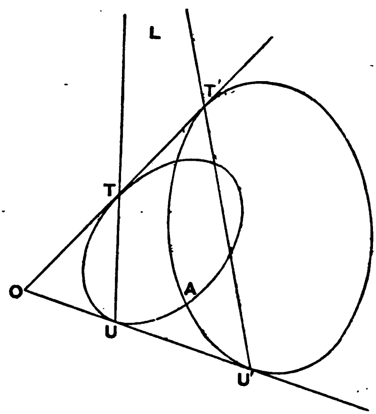
In the first way, every point P on a is homologous with the point P' in which OP cuts β on the same side of MN as P; and in the second way, every point P on a is homologous with the point P'' in which OP cuts β on the opposite side of MN to P.

Ex. 1. *A conic is its own homologue, any point and its polar being c. of h. and a. of h.*

Ex. 2. *If a conic be its own homologue, show that if the c. of h. be given, the a. of h. must be the polar of the c. of h.*

16. *Any two conics are in homology.*

Take any meet O of common tangents TT', UU' as c. of h. Let TU and $T'U'$, the polars of O, cut in L. Let A be one of the four common points of the two conics. Take LA as a. of h. Also take UU' as a pair of corresponding points.



The homologue of the conic TUA can now be found. Suppose the conic TUA to be given by the five points $TTUUA$, where TT are the coincident points in which OT touches the conic, and UU are the coincident points in which OU touches the conic. The homologue of A is A, for A is on the a. of h. The homologues of UU are $U'U'$ by hypothesis. Again, since TT' passes through O, and TU, $T'U'$ meet on the a. of h., T' is the homologue of T, i. e. $T'T'$ are the homologues of TT. Hence the homologues of $TTUUA$ are $T'T'U'U'A$. Hence the homologue of the conic TUA is a conic passing through A and touching OT' at T' and touching OU' at U'; i. e. the homologue of one given conic is the other given conic.

Hence *two conics are homological in twelve ways.*

For we may take as c. of h. O any one of the six meets of common tangents of the two conics. We may then take as A any one of the four common points of the two conics. But this will only give us two possible axes of h. For the point where LA, the a. of h., meets either conic again will be another common point. Hence there are only two positions of LA, when the position of O has been chosen.

Ex. 1. *Show by using the reciprocal solution to the above that we may take any common chord of the conics as a. of h.*

Ex. 2. *Two conics in different planes may be placed in projection by two rotations, viz.* (i) *about the meet of the planes until the planes coincide, and* (ii) *about any common chord of the conics (when placed in one plane).*

Ex. 3. *Show that two homothetic conics have two centres of similitude.*

Viz. the common apexes belonging to the line at infinity.

Ex. 4. *Show by using the circular points that any two conics which have a common focus are in homology, the common focus being the c. of h.*

Ex. 5. *Any conic is homological with any circle whose centre is at a focus of the conic, the focus being the c. of h.*

17. *If two conics touch, they are homological, taking the point of contact as c. of h.*

This follows as a limiting case of the general theorem; *or thus directly.* Through O, the point of contact, draw any chord cutting one conic in P and the other in P'. Take O as c. of h., and the common chord AB, which does not pass through O, as a. of h. Also take P, P' as homologous points. Consider the homologue of the conic determined by $OOPAB$. It is a conic through $OOP'AB$, i.e. it is the other conic.

18. *If the join XX' of any two homologous points cut the a. of h. in U, then $(OXUX')$ is constant, O being the c. of h.*

For take any fixed pair of homologous points AA'. Then AX, $A'X'$ meet on the a. of h., say at L. Hence if AA' cut the a. of h. in N, we have

$$(OXUX') = (OANA') = \text{constant}.$$

This proof fails if $AA'XX'$ all lie on the same line through O. In this case take any pair of homologous points BB' which do not lie on $AA'XX'$, and let OBB' cut the a. of h. in R.

Then $(OXUX') = (OBRB') = (OANA') =$ constant.

Conversely, *if a point X' be taken such that $(OXUX')$ is constant, O being a fixed point and U the meet of OX with a fixed line, then the figures generated by X and X' will be homological, O being the c. of h. and the fixed line the a. of h.*

For if AA', XX' be two pairs of points thus obtained, since $(OANA') = (OXUX')$, it follows that AX, NU, $A'X'$ meet in a point. Hence the join of any two points meets the join of the corresponding points on a fixed line. Hence the figures are homological.

(OU, XX') is called *the parameter* of the homology.

Ex. 1. *If two homologous lines LX and LX' cut the a. of h. in L, show that $L(OXMX')$ is constant, M being any other point on the a. of h.; and conversely, if LA' be determined as the corresponding line to LA by this definition, show that the figures generated by LA and LA' are homological.*

Ex. 2. *If $(OU, XX') = -1$, show that the figure made up of a figure and its homologue is its own homologue.*

This is called *harmonic homology.*

Notice that harmonic homology bears the same relation to ordinary homology as an involution range bears to two homographic ranges on the same line. In fact the figure $(ABC \ldots A'B'C' \ldots)$ is homologous to the figure $(A'B'C' \ldots ABC \ldots)$, if the two figures $(ABC \ldots)$ and $(A'B'C' \ldots)$ are in harmonic homology.

Ex. 3. *In harmonic homology, if the c. of h. be at infinity in a direction perpendicular to the a. of h., then each figure is the reflexion of the other in the a. of h.*

Ex. 4. *In harmonic homology, if the a. of h. be at infinity, then each figure is the reflexion of the other in the c. of h.*

Ex. 5. *If a conic be its own homologue, show that the homology is harmonic, and that the homologue of the line at infinity is halfway between the c. of h. and its polar. Also show that a conic is an ellipse, parabola, or hyperbola, according as the line halfway between any point and its polar cuts the conic in imaginary, coincident, or real points.*

Ex. 6. *Show that two figures in homology reciprocate into two figures in homology, and that the parameters of homology are numerically equal.*

Ex. 7. *The parameter of homology of two homothetic figures is the reciprocal of the ratio of similitude.*

Ex. 8. *The parameter of homology of two figures in parallel homology is the constant ratio of the ordinates.*

Ex. 9. *Keeping the same c. of h., show that the two parameters of homology of two circles are equal but of opposite signs.*

Ex. 10. *Keeping the radical axis as a. of h., show that the two parameters of homology of two circles are equal but of opposite signs.*

Ex. 11. *If the radical axis of two circles be taken as the a. of h., and if the vanishing lines and the radical axis cut the line of centres in $IJ'N$; show that*

$$SI : IN :: r : r', \text{ and } SJ' : J'N :: r' : r.$$

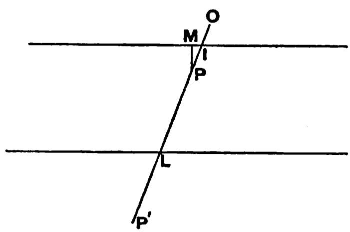
Ex. 12. *If A, B, C be fixed points and P, P' variable points such that $B(APP'C) = A(BPP'C) =$ constant; show that P and P' generate homological figures, of which C is the c. of h. and AB is the a. of h.*

Ex. 13. *Tangents from the point P to a conic meet any line l in L, M, and the other tangents from L, M meet at P'. Show that P and P' generate figures in homology.*

19. *If PP' be any homologous points, and PM the perpendicular from P on the vanishing line of the figure generated by P, then*

$$OP/PM \propto OP',$$

O being the c. of h.



Let OP cut the vanishing line in I and the a. of h. in L. Then, since I corresponds to the point Ω' at infinity upon OP, we have $(OI, PL) = (O\Omega', P'L)$.

Hence $$\frac{OP}{PI} \div \frac{OL}{LI} = \frac{OP'}{P'\Omega'} \div \frac{OL}{L\Omega'} = \frac{OP'}{OL},$$

i. e. $$OP : OP' :: PI : LI :: PM : h,$$

where h is the perpendicular distance between the vanishing line and the a. of h.

Hence $OP : PM :: OP' : h$.

Ex. *Prove the $SP:PM$ property of a focus.*

Form a homologue of the conic, taking S as the c. of h. and the corresponding directrix as vanishing line. Then $SP \div PM \propto SP'$. But by § 12 the locus of P' is a circle with centre S. Hence $SP \div PM$ is constant.

20. *In two homological figures, if (X, p) and (X, q) denote the perpendiculars from the variable point X on two given lines p and q, and if (X', p') and (X', q') denote the perpendiculars from the corresponding point X' in the homologous figure on the corresponding lines p' and q', then* $\dfrac{(X, p)}{(X, q)} \div \dfrac{(X', p')}{(X', q')}$ *is constant.*

For take another point Y, and let XY cut the lines p and q in A and B. Then $X'Y'$ will cut p' and q' in the homologous points A' and B'. Hence, since homological figures are projective, we have

$$(AB, XY) = (A'B', X'Y'),$$

i. e. $\quad AX/AY \div XB/YB = A'X'/A'Y' \div X'B'/Y'B'$,

i. e. $\quad (X, p)/(Y, p) \div (X, q)/(Y, q)$

$$= (X', p')/(Y', p') \div (X', q')/(Y', q').$$

Hence $\quad (X, p)/(X, q) \div (X', p')/(X', q')\quad$ is constant.

Ex. 1. *If X and Y be fixed and p vary, then*
$$(X, p)/(Y, p) \div (X', p')/(Y, p') \textit{ is constant.}$$

Ex. 2. *If i be the vanishing line of the unaccented figure, then $(X, p)/(X, i) \div (X', p')$ is constant.*

Take q' at infinity; then $(X', q') = (Y', q')$.

Ex. 3. *If i and j' be the vanishing lines, then*
$$(X, i) \,.\, (X', j') \textit{ is constant.}$$

Take p and q' at infinity.

Ex. 4. *$OX/(X, p) \div OX'/(X', p')$ is constant.*

Take q and q' as the axis of homology a, and notice that $OX/(X, a) \div OX'/(X', a)$ is constant, since (OU, XX') is constant.

Ex. 5. *$OX/(X, i) \div OX'$ is constant.*

21. *Homographic figures* are defined to be figures in which a point in one figure corresponds to a point in the other figure, a line to a line, the connector of two points to the

connector of two points, the intersection of two lines to the intersection of two lines, and finally a collinear range of points corresponds to a homographic range and a pencil of rays corresponds to a homographic pencil.

Homographic figures exist; for two figures in perspective, whether in homology or in projection, clearly possess the above properties.

No other homographic figures exist; for—

Magnus's theorem. *Two homographic figures can be placed in homology (and therefore in projection).*

Let A, B be the circular points in the first figure, and let A', B' correspond to A, B in the second figure. Place the figures in the same plane, and let AA' and BB' meet at O'. Let O correspond to O', and take any other corresponding points C and C'. First move the first figure without rotation until O comes to O'. This will not alter the positions of A and B; for such a motion does not displace any point at infinity. To prove this, let I be the point at infinity on the parallel lines p and q. Then, after the above motion of translation, p and q are parallel to their former positions and hence still pass through I. Hence I has not moved. Hence AA' and BB' still meet at O', i.e. at O. Now rotate the first figure about O until C is on the line OC'. This will not move A or B. For A and B are the intersections of the line at infinity with any circle with centre at O. But the circle is not changed as a whole by this rotation; and hence its intersections with any fixed curve will remain the same. And the line at infinity is fixed, since parallel lines remain parallel lines and hence still meet on the line at infinity when rotated. Hence AA' and BB' still meet at O. Hence AA', BB', CC' all meet at O. Hence if D, D' are any other corresponding points, DD' passes through O; for $O(ABCD) = O(A'B'C'D')$. Hence the figures are copolar and therefore homological. And we know that homological figures are in projection if rotated about the axis of homology.

Hence homographic figures introduce no new properties

beyond those of figures in perspective; for they can be placed in perspective.

22. *Correlative figures* are defined to be figures such that a point in one figure corresponds to a line in the other and a line to a point; also the connector of two points corresponds to the intersection of two lines and vice versa; and finally a range of points corresponds to a homographic pencil of rays and vice versa.

Correlative figures exist; for clearly reciprocal figures possess the above properties.

Also correlative figures introduce no new properties. For if the figures f_1 and f_2 are correlative, then if we take f_3 reciprocal to f_2, f_1 and f_3 are homographic and therefore projective. For to a point in f_1 corresponds a line in f_2 and to this corresponds a point in f_3; and so on. Hence we can get from f_1 to f_2 by going from f_1 to f_3 by projection, and then from f_3 to f_2 by reciprocation.

MISCELLANEOUS EXAMPLES

1. GENERALIZE by projection and reciprocation the theorems—(1) 'The director circles of all conics inscribed in the same quadrilateral are coaxal,' (2) 'The locus of the centre of an equilateral hyperbola which passes through three given points is a circle.'

2. The portion of a common tangent to two circles α and β between the points of contact is the diameter of the circle γ. If the common chord of γ and α meets that of γ and β in R, show that R is the pole for γ of the line of centres of α and β.

3. Generalize by projection the theorem—'The straight lines which connect either directly or transversely the extremities of parallel diameters of two circles intersect on their line of centres.'

4. A pair of right lines through a fixed point O meet a conic in PQ, $P'Q'$; show that if PP' passes through a fixed point, then QQ' also passes through a fixed point.

5. Generalize by projection and reciprocation the theorem —'A diameter of a rectangular hyperbola and the tangent at either of its extremities are equally inclined to either asymptote.'

6. If P, Q denote any pair of diametrically opposite points on the circumference of a given circle, and QY the perpendicular from Q upon the polar of P with respect to another given circle whose centre is C, show that $QY \cdot CP$ is constant.

What does the theorem become when the circles are orthogonal?

7. Through a given point O draw a line cutting the sides BC, CA, AB of a triangle ABC in points A', B', C', such that $(OA', B'C')$ shall be harmonic.

8. Given the centre of a conic and three tangents, find the point of contact of any one of them.

9. Two similar and similarly situated conics have a common focus which is not a centre of similitude. Prove

that a parabola can be described touching the common chord and the common tangents of the conics, and having its focus at their common focus.

10. Generalize by projection the theorem—'One circle can be described so as to pass through the four vertices of a square and another so as to touch its four sides, the centre of each circle being the intersection of diagonals.'

11. Two conics touch at A, and intersect at B and C. Through O, the point where BC meets the tangent at A, is drawn a chord OPP' of the one conic, and AP, AP' produced if necessary meet the second conic in Q and Q'. Prove that Q, Q' and O are collinear.

12. $ABCD$ is a rectangle, and (AC, PQ), (BD, XY) are harmonic ranges; show that the points P, Q, X, Y lie on a circle.

13. Through O, one of the points of intersection of two circles, the chords POQ and $OP'Q'$ are drawn (P and P' being on one circle and Q and Q' on the other). Show that if $PO : OQ :: OP' : OQ'$, then OP and OP' generate a pencil in involution.

14. O is the orthocentre of the acute-angled triangle ABC. Prove that the polar circles of the triangles OBC, OCA, OAB are orthogonal, each to each.

15. A number of conics are inscribed in a given triangle so as to touch one of its sides at a given point. Show that their points of contact with the other two sides form two homographic ranges which are in perspective.

16. AC, BD are conjugate diameters of a central conic, and P is any point on the arc AB. PA, PB meet CD in Q, R respectively. Prove that the range (QC, DR) is harmonic.

17. Generalize by projection and reciprocation the proposition—'The locus of the foot of the perpendicular upon any tangent to an ellipse from a focus is a circle.'

18. P is the pole of a chord which subtends a constant angle at the focus S of a conic, and SP intersects the chord in Q; find the locus of the point R such that (SR, PQ) is harmonic.

19. A straight line AD is trisected in B, C; the connectors of A, B, C, D, and the point at infinity on AD with

any point S meet another straight line in A', B', C', D', E' respectively; show that $E'B' : E'D' = 3 \, . \, A'B' : A'D'$.

20. From any point Q on a fixed tangent BQ to a circle $AA'B$, straight lines are drawn to A, A', the extremities of a fixed diameter parallel to BQ, meeting the circle again in P, P' respectively; show that the locus of the intersection of $A'P$, AP' is a parabola of which B is the vertex.

21. Two conics α and β intersect in the points A, B, C, D; show that if the pole of AB with regard to α lies on β, then the pole of CD with regard to α lies on β.

If the vertex of a parabola is the pole of one of its chords of intersection with a circle, then another common chord is a diameter of the circle.

22. 'If a circle be drawn through the foci S, H of two confocal ellipses, cutting the ellipses in P and Q, the tangents to the ellipses at P and Q will intersect on the circumference of the circle.'

Generalize this theorem (1) by projection, (2) by reciprocation with respect to the point S, (3) by reciprocation with respect to any point in the plane.

23. 'If two circles of varying magnitude intersect on the side BC of a given triangle ABC and touch AB, AC at B and C respectively; then the locus of O, their other point of intersection, is the circumcircle of the triangle; and the circle on which their centres and the point O lie, always passes through a fixed point.'

Obtain by projection the corresponding theorem when the two circles are replaced (1) by conics, (2) by similar and similarly situated conics.

24. Two ranges are in perspective, and the centre of perspective S is equidistant from the axes of the ranges. The axes are turned about their meet O until they coincide. Show that if S does not coincide with O, an involution is produced; and find the centre and double points.

25. 'If a circle touches two given circles, the connector of its points of contact passes through a centre of similitude of the given circles.'

Reciprocate this proposition with respect to a limiting point.

26. The pairs of points AB, CD form a harmonic range. Prove that, if X is any other point on the same axis, then

the anharmonic ratios (AB, CX) and (AB, DX) are equal and of opposite sign.

27. The connectors of a point D in the plane of the triangle ABC with B, C meet the opposite sides in E, F respectively; show that the triangles BDC, EDF have the same ratio as the triangles ABC, AEF.

28. A, B, C are three points on a straight line; A_1 is the harmonic conjugate of A with respect to BC, B_1 of B with respect to CA, and C_1 of C with respect to AB; show that AA_1, BB_1, CC_1 are three pairs of a range in involution.

29. A conic is reciprocated into a circle. Find the reciprocals of a pair of conjugate diameters.

30. Generalize by projection the theorem—'If a straight line touch a circle and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.'

31. The locus of the pole of a chord of a conic which subtends a right angle at a fixed point is a conic.

32. A quadrilateral $ABCD$ is circumscribed to a conic, and a fifth tangent is drawn at the point P; the diagonals AC, BD meet the tangent at P in α and β, and the points α', β' are taken the harmonic conjugates of α and β with respect to A, C and B, D respectively; show that α', β', P are on a straight line.

33. Through the vertex A of a square $ABCD$ a straight line is drawn meeting the sides BC, CD in E, F. If ED, FB intersect at G, show that CG is at right angles to EF.

34. Determine the envelope of a straight line which meets the sides of a triangle in A, B, C, so that the ratio $AB:AC$ is constant.

35. Generalize by projection the theorem—'If OP, OQ, tangents to a parabola whose focus is S, are cut by the circle on OS as diameter in M and N, then MN will be perpendicular to the axis.'

36. Reciprocate with regard to the focus of the parabola the theorem—'The circle described on a focal radius of a parabola as diameter touches the tangent at the vertex.'

37. The lines joining any point O to four collinear points A, B, C, D cut any transversal through D in A', B', C', D,

Also BC' meets OA in O', $O'B'$ meets AB in B''. Prove that if $(AB, CD) = -1$, then $AB''.CB = 2AC.BB''$.

38. M and N are a pair of inverse points with regard to a given circle whose centre is C. Prove that (1) if P is any point on the circle, $PM^2 : PN^2 :: CM : CN$; (2) if any chord of the circle is drawn through M or N, the product of the distances of its extremities from the straight line bisecting MN at right angles is constant.

39. Points P, Q are taken on the sides AB, AC of a triangle respectively, such that $AP = CQ$; show that the line joining PQ will envelope a parabola.

Through a given point draw a straight line to cut the equal sides AB, AC of an isosceles triangle BAC in P, Q respectively, so that AP is equal to CQ.

40. Given the proposition 'any point P of an ellipse, the two foci, and the points of intersection of the tangent and normal at P with the minor axis are concyclic,' (1) generalize it by projection, (2) reciprocate it with regard to one of the foci.

41. Generalize the following proposition (1) by reciprocating it with respect to A, and (2) by projection—'A fixed circle whose centre is O touches a given straight line at a point A; the locus of the centre of a circle which moves so that it always touches the fixed circle and the fixed straight line is a parabola whose focus is O, and whose vertex is A.'

42. Two circles α and β intersect a conic γ; show that the chords of intersection of α and γ meet the chords of intersection of β and γ in four points which lie on a circle having the same radical axis with α and β.

43. Through any point O in the plane of a triangle ABC are drawn OA', OB', OC' bisecting the supplements of the angles BOC, COA, AOB and meeting BC, CA, AB in A', B', C' respectively; show that the six lines OA, OB, OC, OA', OB', OC' form a pencil in involution.

44. Two conics α and β have double contact at B and C, A being the pole of BC. Tangents from a point X upon AB are drawn to α and β meeting AC in Y and Y'. Show that Y and Y' generate homographic ranges, the double points of which are A and C.

45. A quadrangle $ABCD$ is inscribed in a parabola; through two of its vertices C and D straight lines are drawn parallel to the axis, meeting DA, BC in P and Q; show that PQ is parallel to AB.

46. Prove that the polar reciprocal with regard to a parabola of the circle of curvature at its vertex is a rectangular hyperbola of which the circle is also the circle of curvature at a vertex.

47. The opposite vertices AA', BB', CC' of a quadrilateral circumscribing a conic are joined to a given point O; OA cuts the polar of A in a, OB cuts the polar of B in b, and so on; show that a conic can be drawn through the seven points $Oaa'bb'cc'$.

48. A range on a line is projected from two different vertices on to another line. Find the double points of the projected ranges.

49. If four points A, B, C, D be taken on the circumference of a circle, prove that the centres of the nine-point circles of the four triangles ABC, BCD, CDA, DAB will lie on the circumference of another circle, whose radius is one-half that of the first.

50. If the orthocentre of a triangle inscribed in a parabola be on the directrix, then the polar circle of the triangle passes through the focus.

51. A and BC are a given pole and polar with regard to a conic; DE is a given chord through A; P, Q, R, ... are any number of points on the conic, and P', Q', R', ... are the points where EP, EQ, ER, ... meet BC. Prove that $D(PP', QQ', RR', \ldots)$ is an involution; and determine its double lines.

52. $ABCD$ is a quadrilateral circumscribing a conic α. AB, DC meet in E, and BC, AD in F, and a conic β is drawn through the points B, D, F, E. Prove that the four tangents to α at the points where the conics intersect pass two and two through the pair of points where AC cuts β.

53. Two conics α, β intersect in the points A, B, C, D. If the pole of AB with respect to α coincides with the pole of CD with respect to β, prove that the pole of CD with respect to α will coincide with the pole of AB with respect to β.

54. Three conics all pass through the same two points A, B. The first and second conics intersect one another in two other points C, D; and the pole of AB with regard to the second conic lies on the first conic. The third conic touches the line joining C, D; and the pole of AB with regard to it lies on the second conic. Show that the tangents, other than CD, drawn from the points C, D to the third conic meet on the circumference of the first conic.

55. Given the asymptotes of a conic and another tangent, show how to construct the pair of tangents from a given point to the conic.

Given the three middle points of the sides of a given triangle, draw a straight line through a given point to bisect the triangle.

56. A conic cuts the sides of a triangle ABC in the pairs of points $a_1\ a_2$, $b_1\ b_2$, $c_1\ c_2$ respectively; if Bb_2, Cc_2 intersect in α_1, and Bb_1, Cc_2 in α_2, and so on, and if $\beta_1\ \beta_2\ \beta_3\ \beta_4$, $\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4$ be similarly constructed; show that the straight lines obtained by putting in various suffixes in $A\alpha$, $B\beta$, $C\gamma$ meet, three by three, in eight points.

57. Reciprocate the proposition that the nine-point circle of a triangle touches the inscribed circle (1) with regard to one of the angular points of the triangle, (2) with regard to the middle point of one of its sides.

58. 'If, from a point within a circle, more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.'

Generalize the above proposition (1) by reciprocation, (2) by projection.

59. TP and TQ are tangents of a conic and PQ is bisected in V; also TV is bisected by the curve. Show that the conic is a parabola.

60. A conic of constant eccentricity is drawn with one focus at the centre of a given circle and circumscribing a triangle self-conjugate with respect to the given circle; show that the corresponding directrices for different positions of the triangle will envelope a circle.

61. A straight line PQ moves so as to make upon two fixed straight lines intercepts AP and AQ whose difference is constant; prove that it will always touch a fixed parabola, and construct the focus and directrix of the parabola.

62. By reciprocation deduce a proposition relating to the circle from the following—'The locus of a point dividing in a given ratio the ordinate PN of a parabola is another parabola having the same vertex and axis.'

63. The envelope of a straight line which moves so that two fixed circles intercept on it chords of equal length is a parabola.

64. Given a conic and a pair of straight lines conjugate with regard to it, project the conic into a parabola of which the projections of the given lines shall be latus rectum and directrix.

65. An ellipse has the focus of a parabola for centre and has with it contact of the third order at its vertex. Tangents are drawn to the two conics from any point on their common tangent, and the harmonic conjugate of this latter with regard to them is taken. Prove that its envelope is the common circle of curvature of the two conics at the common vertex.

66. ABC, DEF are two triangles inscribed in a conic. BC, CA, AB are parallel respectively to EF, FD, DE. Prove that AD, BE, CF are diameters of the conic.

67. Find the double rays of the pencils $O(ABC\ldots)$ and $O(A'B'C'\ldots)$, each of which is in perspective with the pencil $V(A''B''C''\ldots)$.

68. $ABCD$ is a quadrangle, and P, Q the two diagonal points which do not lie on AB. Two conics are drawn, the first through A, B, C, D, the second through A, B, P, Q. Prove that, if R is a point on the second conic, and if AR, BR meet the first conic at C', D' respectively, then $C'B$, $D'A$ will also meet on the second conic.

69. Through a point O is drawn a straight line cutting a conic in AB, and on AB are taken points CD, such that

$$(1 \div OC) + (1 \div OD) = (1 \div OA) + (1 \div OB).$$

Then if MN be the points of contact of tangents from D, and LR those of tangents from C, show that either LM and RN, or LN and RM, meet in O.

70. Construct the conic which passes through the four points $ABCD$ and is such that AB and CD are conjugate lines with regard to it.

71. AOB and COD are two diameters of a circle and QR

is a chord parallel to AB; if P be the intersection of CQ and DR, or of DQ and CR, and if from P be drawn PM parallel to AB to meet CD in M, then $OM^2 = OD^2 + PM^2$.

72. AB, AC are two chords of an ellipse equally inclined to the tangent at A; show that the ratio of the chords is the duplicate of the ratio of the diameters parallel to them.

73. Construct, by means of the ruler only, a conic which shall pass through two given points and have a given self-conjugate triangle.

Also construct the pole of the connector of the given points with respect to the conic.

74. Through a fixed point A any two straight lines are drawn meeting a conic in B, B' and C, C' respectively; parallels through A to BC', $B'C$ meet $B'C$, BC' respectively in D, E; find the locus of D and of E.

75. Two equal tangents TP and TQ of a parabola are cut in M and N by a third tangent; show that $TM = QN$.

76. The tangents at two points of an ellipse, whose foci are S, H, meet in T, and the normals at the same points meet in O; prove that the perpendiculars through S, H to ST, HT respectively divide OT harmonically.

Deduce a construction for the centre of curvature at any point of the ellipse.

77. An ellipse may be regarded as the polar reciprocal of the auxiliary circle with respect to an imaginary circle of which a focus is the centre. Prove this, and find the lines which correspond to the centre and the other focus of the ellipse.

78. Two conics u, v intersect in A, B, C, D; E, F are the poles of CD with regard to the conics u, v respectively, and AE, AF meet CD in G, H respectively; a straight line is drawn through A meeting u, v in P, Q respectively; show that the locus of the intersection of PH, QG is a straight line passing through B and through the intersection of EF, CD.

79. Show that two triangles, one inscribed in and the other escribed to a given triangle, and both in perspective with it, are in perspective.

Each of the triangles determined by the common tangents of two conics is in perspective with each of the triangles determined by the common points of the conics.

80. Two circles cut each other orthogonally; show that the distances of any point from their centres are in the same ratio as the distances of the centre of each circle from the polar of the point with respect to the other.

The directrix of a fixed conic is the polar of the corresponding focus with respect to a fixed circle; with any point on the conic as centre a variable circle is described cutting the fixed circle orthogonally; find the envelope of the polar of the focus with respect to the variable circle.

81. Obtain a construction for projecting a conic and a point within it into a parabola and its focus.

82. A conic circumscribes a triangle ABC, the tangents at the angular points meeting the opposite sides on the straight line DEF. The lines joining any point P on DEF to A, B, C meet the conic again in A', B', C'. Show that the triangle $A'B'C'$ envelopes a fixed conic inscribed in ABC, and having double contact with the given conic at the points where it is met by DEF. Show also that the tangents at A', B', C' to the original conic meet $B'C'$, $C'A'$, $A'B'$ in points lying on DEF.

83. $ABCD$ is a quadrilateral whose sides AB, CD meet in E, and AD, BC in F; A is a fixed point, EF a fixed straight line, and B, C lie each upon one of two fixed straight lines concurrent with EF; find the locus of D.

84. All the tangents of a conic are inverted from any point. Show that the locus of the centres of all the circles into which they invert is a conic.

85. If A, B, C, D, be four collinear points, and O any point whatever, prove that $\Sigma\{OA^2 \div (AB \,.\, AC \,.\, AD)\} = 0$.

Also show that if A', B', C', D' be four concyclic points, then $\Sigma\{1 \div (A'B' \,.\, A'C' \,.\, A'D')\} = 0$, the sign of any rectilinear segment being the same as in the preceding identity.

86. If O be the intersection of the common tangents to two conics having double contact, and if a straight line through O meet the two conics in P, P' and Q, Q' respectively, prove that

$$PQ \,.\, P'Q' \,.\, (PO + P'O) = PO^2 \,.\, P'Q' + P'O^2 \,.\, PQ,$$

and that $$PQ \,.\, PQ' : P'Q \,.\, P'Q' :: PO^2 : P'O^2.$$

87. Describe a conic to touch a given straight line at a given point and to osculate a given circle at a given point.

88. If a system of conics have a common self-conjugate

triangle, any line through one of the vertices of this triangle is cut by the system in involution.

Two conics, U and U', touch their common tangents in $ABCD$ and $A'B'C'D'$; show that AB cuts U, U' and the other sides of the quadrilateral of tangents in six points in involution.

89. Four points A, B, C, D are taken on a conic such that AB, BC, CD touch a conic having double contact with it; show that A and D generate homographic ranges on the conic, and find the common points of the ranges.

90. The angular points ABC of a triangle are joined to a point O and the bisectors of the angles BOC, COA, AOB meet the corresponding sides of the triangle in $\alpha_1\ \alpha_2$, $\beta_1\ \beta_2$, $\gamma_1\ \gamma_2$; show that these points lie three by three on four straight lines; and that if O lie on the circle circumscribing the triangle, each of the lines $\alpha_1\ \beta_2\ \gamma_2$, &c., passes through the centre of a circle touching the three sides of the triangle.

91. 'If from a point T on the directrix of a parabola whose vertex is A tangents TP, TQ are drawn to the curve, and PA, QA joined and produced to cut the directrix in M, N, then will T be the middle point of MN.'

Obtain from the above theorem by reciprocation a property of (1) a circle, (2) a rectangular hyperbola.

92. In two figures in homology M and M' are homologous points and O is the centre of perspective. Show that OM is to MM' as the perpendicular from M on its vanishing line is to the perpendicular from M on the axis of perspective.

93. Given two points A, B on a rectangular hyperbola and the polar of a given point O in the line AB; determine the points of intersection of the curve with the straight line drawn through O perpendicular to AB.

94. Show how to project a given quadrilateral into a quadrilateral $ABCD$ such that AB is equal to AC, and that D is the centre of gravity of the triangle ABC.

95. A circle has double contact with an ellipse, and lies within it. A chord of the ellipse is drawn touching the circle, and through its middle point is drawn a chord of the ellipse parallel to the minor axis. Show that the rectangle contained by the segments of this chord is equal to the

rectangle contained by the segments into which the first is divided by the point of contact.

96. *ABCDEF* is a hexagon inscribed in one conic and circumscribing another. The connectors of its vertices with any point O in its plane meet the former conic again in the vertices of a second hexagon $A'B'C'D'E'F'$. Prove that it is possible in this to inscribe another conic.

97. $ABCD$, $AB'C'D'$ are two parallelograms having a common vertex A and the sides AB, AD of the one along the same straight lines as the sides AB', AD' respectively of the other. Show that the lines BD', $B'D$, CC' are concurrent.

98. Three conics α, β, γ are inscribed in the same quadrilateral. From any point, tangents a, b are drawn to α, and tangents a', b' to β. Show that if a, a' are conjugate lines with respect to γ, so are b, b'.

99. If three tangents to a conic can be found such that the circle circumscribing the triangle formed by them passes through a focus, the conic must be a parabola.

100. From each point on a straight line parallel to an axis of a conic is drawn a straight line perpendicular to the polar of the point; show that the locus of the foot of the perpendicular is a circle.

101. Give a construction for projecting a conic into another, of which the projections of three given points shall be a focus, an extremity of the minor axis, and a vertex.

102. Find the locus of the centre of a circle which divides two given segments of lines harmonically.

103. The sides AB, AD of a parallelogram $ABCD$ are fixed in position, and C moves on a fixed line; show that the diagonal BD envelopes a parabola.

104. A tangent of a hyperbola whose centre is C meets the asymptotes in P and Q; show that the locus of the orthocentre of the triangle CPQ is another hyperbola.

105. Through fixed points A and B are drawn conjugate lines for a given conic. Show that the locus of their meet is the conic through A, B and the points of contact of tangents from A and B.

106. A, B, C, D are four points on a conic, and O is the pole of AB. Show that $O(AB, CD)$ is the square of (AB, CD).

107. A, B, C, D are four points on a conic. The tangent at A meets BC, CD in a_1, a_2; the tangent at B meets CD, DA in b_1, b_2; and so on. Show that the eight points a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 lie on a conic.

108. The centre O of a conic lies on the directrix of a parabola, and a triangle can be drawn which is circumscribed to the parabola and self-conjugate for the conic. Show that the tangents from O to the parabola are the axes of the conic.

109. Two sides AQ, AR of a triangle AQR circumscribed to a given circle are given in position; the circles escribed to AQ and AR touch AQ and AR in V and U; show that the locus of the meet of QU and RV is a hyperbola with AQ and AR as asymptotes.

110. If the chords OP, OQ of a conic are equally inclined to a fixed line; then, if O be a fixed point, PQ passes through a fixed point.

111. A fixed line l meets one of the system of conics through the four points A, B, C, D in P and Q; show that the conic touching AB, CD, PQ and the tangents at P and Q touches a fourth fixed line.

112. Triangles can be inscribed in α which are self-conjugate for β; ABC is a triangle inscribed in α and $A'B'C'$ is its reciprocal for β; show that the centre of homology of ABC and $A'B'C'$ is on α.

113. Six circles of a coaxal system touch the sides of a triangle ABC inscribed in any coaxal in the points aa', bb', cc'; show that these points are the opposite vertices of a quadrilateral.

114. A, B, C, D are four points on a circle, and A', B', C', D' are the orthocentres of the triangles BCD, CDA, DAB, ABC. Show that the figures $ABCD$, $A'B'C'D'$ are superposable.

115. Any conic α which divides harmonically two of the diagonals of a quadrilateral is related to any conic β inscribed in the quadrilateral in such a way that triangles can be inscribed in α which are self-conjugate for β.

116. The envelope of the axes of all conics touching four tangents of a circle is a parabola.

117. If $(AA', BB') = -1 = (AA', PQ) = (BB', PQ')$; show that (AA', BB', QQ') is an involution.

118. If two conics can be drawn to divide four given

segments harmonically, then an infinite number of such conics can be drawn.

119. If (AA', BB', CC') be an involution, show that
$$(A'A, BC') + (B'B, CA') + (C'C, AB') = 1.$$

120. T is a point on the directors of the conics a and β. The reciprocal of a for β meets the polar of T for β in Q, R. Show that the angle QTR is right.

121. Through the centre O of a circle is drawn a conic, and A and A' are a pair of opposite meets of common tangents of the circle and conic; show that the bisectors of the angle AOA' are the tangent and the normal at O.

122. A given line meets one of a series of coaxal circles in P, Q. The parabola which touches the line, the tangents at P, Q, and the radical axis has a third fixed tangent.

123. If a series of conics be inscribed in the same quadrilateral of which A, A' is a pair of opposite vertices, and if, from a fixed point O, tangents OP, OQ be drawn to one of the conics, the conic through $OPQAA'$ will pass through a fourth fixed point.

124. On a tangent to a circle inscribed in a triangle ABC are taken points a, b, c, such that the angles subtended by Aa, Bb, Cc at the centre O are equal; show that Aa, Bb, Cc are concurrent.

125. Through two given points, four conics can be drawn for which three given pairs of lines are conjugate; and the common chord is divided harmonically by every conic through its four poles for the conics.

126. The locus of the pole of a common chord of two conics, for a variable conic having double contact with the two given conics, consists of a conic through the two common points on the given chord, together with the join of the poles of the chord for the two conics.

127. Find the locus of the centre of a conic which passes through two given points and divides two given segments harmonically.

128. A variable conic passes through three fixed points and is such that triangles can be inscribed in it which are self-polar for a given conic. Show that it passes through a fourth fixed point.

129. If a variable conic touch three fixed lines, and be such that triangles can be drawn circumscribing it which are self-polar for a given conic, then the variable conic will have a fourth fixed tangent, and the chords of contact of the variable conic with the fixed lines pass through fixed points.

130. The directrix of a parabola which has a fixed focus and is such that triangles can be described about it which are self-polar for a given conic, passes through a fixed point.

131. A conic U passes through two given points and is such that two sets of triangles can be inscribed in it, one self-polar for a fixed conic V and the other self-polar for a fixed conic W. Show that U has a fixed self-polar triangle.

132. A variable conic U cuts a given conic V in two given points and also touches it and is such that triangles can be inscribed in it self-polar for a given conic W. Show that U touches another fixed conic.

133. Three parabolas are drawn, two of which pass through the four points common to two conics and the third touches their common tangents. Show that their directrices are concurrent.

134. If a system of rectangular hyperbolas have two points common, any line perpendicular to the common chord meets them in an involution.

135. The reciprocal of a circle through the centre of a rectangular hyperbola, taking the r. h. itself as base conic, is a parabola whose focus is at the centre of the r. h.

136. The reciprocal of any circle, taking any r. h. as base conic, is a conic, one of whose foci is at the centre of the r. h.; and the centre of the circle reciprocates into the corresponding directrix.

137. The chords AB and $A'B'$ of a conic α meet in V. β is the conic touching AB, $A'B'$ and the tangents at A, B, A', B'. VL and VL' divide AVA' harmonically and cut the conic α in LM and $L'M'$. Show that the other joins of the points L, M, L', M' touch β. Also any tangent of β meets AB and $A'B'$ in points which are conjugate for α.

138. The director circle of a conic is the conic through the circular points and the points of contact of tangents from these points to the conic.

139. Tangents to a circle at P and Q meet another circle in AB and CD; show that a conic can be drawn with a focus at either limiting point of the two circles and with PQ as corresponding directrix, which shall pass through $ABCD$.

140. Tangents to a conic from two points PP' on a confocal meet again in the opposite points QQ' and RR'. Show that QQ' lie on one confocal and RR' on another; and that the tangents to the confocals at $PP'QQ'RR'$ are concurrent.

141. The centroid of the meets of a parabola and a circle is on the axis of the parabola.

142. A variable tangent of a circle meets two fixed parallel tangents in P and Q, and a fixed line through the centre in R. X is taken so that $(PQ, RX) = -1$. Show that the locus of X is a concentric circle.

143. A triangle is reciprocated for its polar circle. Show that the reciprocal of the centroid is the radical axis of the circumcircle and the nine-point circle.

144. The reciprocal of a triangle for its centroid is a triangle having the same centroid.

145. Triangles can be circumscribed to α which are self-conjugate for β. A tangent of α cuts β in P and Q; and a conic γ is drawn touching β at P and at Q. Show that triangles can be circumscribed to α which are self-conjugate for γ.

146. PP' is a chord of a parabola. Any tangent of the parabola cuts the tangent parallel to PP' in X, and the tangents at P and P' in R and R'; show that $RX = XR'$.

147. If the conic α be its own reciprocal for the conic β, then β is its own reciprocal for α.

148. Given a conic α and a chord BC of α, a conic β can be found having double contact with α at B and C, such that α is its own reciprocal for β.

149. A conic cannot be its own reciprocal for a conic having four-point contact with it.

150. If the conic α be its own reciprocal for the conic β, then α and β can be projected into concentric circles, the squares of whose radii are numerically equal.

151. Any point P on a conic and the pole of the normal at P are conjugate points for the director circle.

152. The pole of the normal at any point P of a conic is the centre of curvature of P for the confocal through P.

153. ABC is a triangle, and AL, BM, CN meet in a point, LMN being points on BC, CA, AB. Three conics are described, one touching BM, CN at M, N and passing through A; so the others. Prove that at A, B, C respectively they are touched by the same conic.

154. The lines joining four fixed points in a plane intersect in pairs in points $O_1O_2O_3$, and P is a variable point. Prove that the harmonic conjugates of O_1P, O_2P, O_3P for the pairs of lines meeting in $O_1O_2O_3$ respectively, intersect in a point.

155. If a parabola touch the sides of a fixed triangle, the chords of contact will each pass through a fixed point.

156. The six intersections of the sides of two similar and similarly situated triangles lie on a conic, which is a circle if the perpendicular distances between the pairs of parallel sides are proportional to the sides of the triangle.

157. Two conics have double contact, O being the intersection of the common tangents. From P and Q on the outer conic pairs of tangents are drawn to the inner, forming a quadrilateral, and R is the pole of PQ with respect to the inner conic. Prove that two diagonals of the quadrilateral pass through R, and that one of these diagonals passes through O.

158. A conic is drawn through the middle points of the lines joining the vertices of a fixed triangle to a variable point in its plane, and through the points in which these joining lines cut the sides of the triangle. Determine the locus of the variable point when the conic is a rectangular hyperbola; and prove that the locus of the centre of the rectangular hyperbola is a circle.

159. The feet of the normals from any point to a rectangular hyperbola form a triangle and its orthocentre.

160. ABC is a triangle and $A'B'C'$ are the middle points of its sides. O is the orthocentre. AO, BO, CO meet the opposite sides in DEF. EF, FD, DE meet the sides in LMN. Prove that OL is perpendicular to AA', OM to BB', and ON to CC'.

161. A variable conic touches the sides AB, AC of a triangle ABC at B and C. Prove that th points of contact

of tangents from a fixed point P to the conic lie on a fixed conic through $PABC$.

162. Given two tangents to a parabola and a fixed point on the chord of contact, show that a third tangent is known.

163. Tangents to a conic from two points on a confocal form a quadrilateral in which a circle can be inscribed.

164. AA', BB', CC' are the opposite vertices of a quadrilateral formed by four tangents to a conic. Three conics pass respectively through AA', BB', CC' and have three-point contact with the given conic at the same point P. Show that the poles of AA', BB', CC' with respect to the conics through AA', BB', CC' respectively coincide, and the four conics have another common tangent.

165. If two conics, one inscribed in and the other circumscribed to a triangle, have the orthocentre as their common centre, they are similar, and their corresponding axes are at right angles.

166. A fixed tangent is drawn to an ellipse meeting the major axis in T. QQ' are two points on the tangent equidistant from T. Show that the other tangents from Q and Q' to the ellipse meet on a fixed straight line parallel to the major axis.

167. With a fixed point P as focus a parabola is drawn touching a variable pair of conjugate diameters of a fixed conic. Prove that it has a fixed tangent parallel to the polar of P.

168. A conic is described having one side of a triangle for directrix, the opposite vertex for centre, and the orthocentre for focus; prove that the sides of the triangle which meet in the centre are conjugate diameters.

169. The radius of curvature in a rectangular hyperbola is equal to half the normal chord.

170. The radius of curvature in a parabola is equal to twice the intercept on the normal between the directrix and the point of intersection of the normal and the parabola.

171. Two ellipses touch at A and cut at B and C. Their common tangents, not at A, meet that at A in Q and R and intersect in P. Prove that BQ and CR meet on AP, and so do BR and CQ.

172. A transversal is drawn across a quadrangle so that

the locus of one double point of the involution determined on it is a straight line. Show that the locus of the other is a conic circumscribing the harmonic triangle of the quadrangle.

173. PQ is a chord of one conic α and touches another conic β. Prove that P, Q are conjugate for a third conic γ.

174. XYZ is a triangle self-conjugate for a circle. The lines joining XYZ to a point D on the circle meet the circle again in A, B, C respectively. Show that as D varies, the centre of mean position of $ABCD$ describes the nine-point circle of XYZ.

175. Two conics are described touching a pair of opposite sides of a quadrilateral, having the remaining sides as chords of contact, and passing through the intersection of its diagonals; show that they touch at this point.

176. With a given point O as focus, four conics can be drawn having three given pairs of points conjugate; and the directrices of these conics form a quadrilateral such that the director circles of all the inscribed conics pass through O.

177. The line joining two points A and B meets two lines OQ, OP in Q and P. A conic is described so that OP and OQ are the polars of A and B with regard to it. Show that the locus of its centre is the line OR where R divides AB so that $AR : RB :: QR : RP$.

178. A chord of a conic passes through a fixed point. Prove that the other chord of intersection of the conic and the circle on this chord as diameter passes through a fixed point.

179. One of the chords of intersection of a circle and a r. h. is a diameter of the circle. Prove that the opposite chord is a diameter of the r. h.

180. Tangents are drawn to a conic α parallel to conjugate diameters of a conic β. Prove that they will cut on a conic γ, concentric with α and homothetic with β. Also γ will meet α in points at which the tangents to α are parallel to the asymptotes of β.

181. Four concyclic points are taken on a parabola. Prove that its axis bisects the diagonals of the quadrilateral formed by the tangents to the parabola at these points.

182. If four points be taken on a circle, the axes of the

two parabolas through them are the asymptotes of the centre-locus of conics through them.

183. The locus of the middle point of the intercept on a variable tangent to a conic by two fixed tangents is a conic having double contact with the given one where it is met by the diameter through the intersection of the fixed tangents.

184. On two parallel straight lines fixed points A, B are taken and lengths AP, BQ are measured along the lines, such that $AP+BQ$ is constant. Show that AQ and BP cut on a fixed parabola.

185. Chords AP, AQ of a conic are drawn through the fixed point A on the conic, such that their intercept on a fixed line is bisected by a fixed point. Show that PQ passes through a fixed point.

186. Three tangents are drawn to a fixed conic, so that the orthocentre of the triangle formed by them is at one of the foci; prove that the polar circle and circumcircle are fixed.

187. Given four straight lines, show that two conics can be constructed such that an assigned straight line of the four is directrix and the other three form a self-polar triangle; and that, whichever straight line be taken as directrix, the corresponding focus is one of two fixed points.

188. Parallel tangents are drawn to a given conic, and the point where one meets a given tangent is joined to the point where the other meets another given tangent. Prove that the envelope of the joining line is a conic to which the two tangents are asymptotes.

189. With a point on the circumcircle of a triangle as focus, four conics are described circumscribing the triangle: prove that the corresponding directrices will pass each through a centre of one of the four circles touching the sides.

190. Three conics are drawn touching each pair of the sides of a triangle at the angular points where they meet the third side and passing through a common point. Show that the tangents at this common point meet the corresponding sides in three points on a straight line, and the other common tangents to each pair of conics pass respectively through these three points.

191. $ABCD$ is a quadrilateral circumscribing a conic, and through the pole O of AC a line is drawn meeting CD, DA, AB, BC, and CA in $PQRST$ respectively. Show that PQ, RS subtend equal angles at any point on the circle whose diameter is OT.

192. The normal at a fixed point P of an ellipse meets the curve again in Q, and any other chord PP' is drawn; QP' and the straight line through P perpendicular to PP' meet in R; prove that the locus of R is a straight line parallel to the chord of curvature of P and passing through the pole of the normal at P.

193. Two tangents of a hyperbola, a, are asymptotes of another conic, β. Prove that if β touch one asymptote of a, it touches both.

194. A conic is drawn through four fixed points $ABCD$. BC, AD meet in A'; CA, BD in B'; AB, CD in C'; and O is the centre of the conic. Prove that $\{ABCD\}$ on the conic $= \{A'B'C'O\}$ on the conic which is the locus of O.

195. Tangents drawn to a conic at the four points $ABCD$ form a quadrilateral whose diagonals are aa', bb', cc' (the tangents at ABC forming the triangle abc and being met by the tangent at D in $a'b'c'$). The middle points of the diagonals are $A'B'C'$, and the centre is O. Prove that $\{A'B'C'O\} = \{ABCD\}$ at any point of the conic.

196. If a right line move in a plane in any manner, the centres of curvature at any instant of the paths of all the points on it lie on a conic.

197. Defining a bicircular quartic as the envelope of a circle which moves with its centre on a fixed conic so as to cut orthogonally a fixed circle, show that it is its own inverse with respect to any one of the vertices of the common self-conjugate triangle of the fixed circle and conic, if the radius of inversion be so chosen that the fixed circle inverts into itself.

198. A quadrilateral is formed by the tangents drawn from two fixed points on the radical axis of a system of coaxal circles to any circle of the system. Prove that the locus of one pair of opposite vertices is one conic, and of the remaining pair is another conic, and the two fixed points are the foci of both these conics.

199. Two fixed straight lines through one of the foci of

a system of confocal conics meet any one of the conics in PP', QQ'. Prove that the envelope of PQ and $P'Q'$ is one parabola, and of PQ', $P'Q$ is another parabola. Also the points of contact of PQ, $P'Q'$, $P'Q$, PQ' with their respective envelopes lie on a straight line parallel to the conjugate axis of the system, which axis touches both parabolas.

200. A parallelogram with its sides in fixed directions circumscribes a circle of a coaxal system. Prove that the locus of one pair of opposite vertices is one conic and of the remaining pair is another conic, and the common tangents of these two conics are the parallels through the common points of the system to the sides of the parallelogram. Prove also that the tangents at the vertices of any such parallelogram to their respective loci meet in a point on the line of centres of the system.

201. O is the centre of a conic circumscribing a triangle, and O' is the pole of the triangle for this conic. Show that O is the pole of the triangle for that conic which circumscribes the triangle and has its centre at O'.

202. AA', BB', CC' are the three pairs of opposite vertices of a quadrilateral. A conic through BB', CC' and any fifth point P meets AA' in X and Y. Prove that PX, PY are the double lines of the involution $P\{AA', BB', CC'\}$.

203. If tangents be drawn to a system of conics having four common tangents, from a fixed point (X) on a side (AA') of the self-conjugate triangle of the system, the points of contact will lie on a conic (viz. $XBB'CC'$).

204. AA', BB', CC' are the three pairs of opposite vertices of a quadrilateral. A straight line meets AA', BB', CC' in LMN. Prove that the conics $LBB'CC'$, $MCC'AA'$, $NAA'BB'$, and the conic touching the sides of the quadrilateral and also LMN, have a common point.

205. Three conics have double contact at the same two points, and A, B, C are their centres. A straight line parallel to ABC meets them in PP', QQ', RR' respectively, and O is any point on this straight line. Prove that

$$OP \cdot OP' \cdot BC + OQ \cdot OQ' \cdot CA + OR \cdot OR' \cdot AB = 0.$$

206. In XXVIII, end, Ex. 2, prove that if O' be the second fixed point, then CO, CO' are equally inclined to the axes, and $CO \cdot CO' = CS^2$.

207. If triangles can be inscribed in a conic a and cir-

cumscribed to a conic β, the locus of the centroid of such a triangle is a conic homothetic with a.

208. If the conic β be a parabola, this locus is a straight line.

209. This straight line is parallel to the line joining points on the parabola where the tangents are parallel to the asymptotes of a.

210. The tangents at three points of a rectangular hyperbola form a triangle, of which the circumcircle has its centre at a vertex and passes through the centre of the hyperbola. Show that the centroid of the three points lies on the conjugate axis.

211. Show that the orthocentre of the three points in Ex. 210 is the vertex which is the centre of the circle.

212. If in Ex. 207 the conics a and β are homothetic, the centroid of the three points of contact with β of such a triangle is a fixed point.

213. If the conics a and β are coaxial, then the normals to a at the vertices of any such triangle are concurrent and also the normals to β at the points of contact of the sides; and conversely, if PQR be three points on a conic such that the normals at these points are concurrent, a coaxial conic can be inscribed in the triangle PQR.

214. If the conics a and β are both parabolas, the locus of the centroid is parallel to the axis of a.

215. If a and β are parabolas with the same axis, whose latera recta are l and l', then $l' = 4\,l$.

216. Given a triangle self-conjugate for a conic, if a directrix touch a conic β inscribed in the triangle, then the corresponding focus lies on the director circle of β.

217. A conic is inscribed in a triangle self-conjugate for a rectangular hyperbola, with one focus on the hyperbola. Show that its major axis touches the hyperbola.

218. A triangle is inscribed in a conic and circumscribed to a parabola. Prove that the locus of the centre of its circumscribing circle is a straight line.

219. The following pairs of conics are manifoldly related—

(i) A rectangular hyperbola, and a parabola whose focus

is at the centre of the r. h. and whose directrix touches the r. h.

(ii) Two rectangular hyperbolas, each passing through the centre of the other and having the asymptotes of one parallel to the axes of the other.

220. If the polar circle of three tangents to a conic passes through a focus, the orthocentre lies on the corresponding directrix.

221. If a triangle inscribed in a parabola has its orthocentre on the directrix, its polar circle passes through the focus.

222. A circle has its centre on the directrix and touches the sides of a triangle self-conjugate for a parabola. Show that it passes through the focus.

223. Triangles can be inscribed in a conic α so as to be self-conjugate for a conic β. A circle has double contact with α along a tangent to β. Show that it cuts orthogonally the director of β.

224. Two conics, in either of which triangles can be inscribed self-conjugate for a third conic, have double contact. Show that their chord of contact touches this conic.

225. From any point P two tangents PQ, PR are drawn to an ellipse: if C is the centre of the ellipse, then all hyperbolas drawn through P and C and having their asymptotes parallel to the axes of the ellipse cut QR harmonically.

226. A conic circumscribes a triangle self-conjugate for a parabola and has its centre on the parabola. Show that an asymptote touches the parabola.

227. A circle through the centre of a rectangular hyperbola cuts it in $ABCD$. Show that the circle circumscribing the triangle formed by the tangents to the r. h. at ABC passes through the centre of the hyperbola, and has its centre on the hyperbola at the extremity D' of the diameter through D; and D' is the orthocentre of ABC.

228. Show that if D be the pole of the triangle ABC for a conic, then A, B, C are the poles of the triangles BCD, ACD, ABD respectively. *Such a quadrangle may be said to be self-conjugate for the conic.*

229. If triangles can be inscribed in β which are self-

conjugate for a, then quadrangles can be inscribed in β which are self-conjugate for a; and conversely.

230. If triangles can be circumscribed to β which are self-conjugate for a, then quadrilaterals can be circumscribed to β which are self-conjugate for a; and conversely.

231. If we can describe triangles to touch a conic a and to be self-polar for each of two conics β and γ, then the four intersections of β and γ form a self-polar quadrangle for a.

232. If triangles can be inscribed in each of two conics β, γ so as to be self-polar for a conic a, then triangles self-polar for a can be inscribed in any conic through the intersections of β and γ.

233. If triangles can be circumscribed to each of two conics β, γ self-polar for a conic a, then triangles self-polar for a can be circumscribed to any conic touching the common tangents of β and γ.

234. The polars of a fixed triangle for a system of four-point conics envelope a conic touching the sides of the triangle.

235. The poles of a fixed triangle for a system of conics having four common tangents lie on a conic circumscribing the triangle.

236. If the system of four-tangent conics is a system of confocals, the locus of the poles is a rectangular hyperbola.

237. If two conics are manifoldly related, and the first passes through the centre of the second, then the second passes through the centre of the first.

238. Three tangents to a conic, a, form a triangle. A conic, β, circumscribes the triangle and passes through the centre of a and the pole of the triangle with respect to a. Prove that its centre lies on a.

239. A rectangular hyperbola circumscribes a triangle and passes through the centre of one of the circles touching the sides. Show that its centre lies on this circle.

240. Hence prove Feuerbach's theorem, viz.—the nine-point circle of any triangle touches the inscribed and escribed circles.

241. Show that in Ex. 239 the poles of the triangle for these circles lie on the respective hyperbolas; and the polars of the triangle for the hyperbolas are tangents to the respective circles.

242. The nine-point circle of a triangle inscribed in a rectangular hyperbola touches the polar-circle of the triangle formed by the tangents at the vertices, at the centre of the conic.

243. The pole with respect to a parabola of the triangle formed by three tangents to it lies on the minimum ellipse circumscribing the triangle.

244. The polar in this case passes through the centroid of the triangle.

245. The pole with respect to a parabola of an inscribed triangle lies on the maximum ellipse inscribed in the triangle.

246. The two conics in the last example are reciprocal with respect to a conic with its centre at this pole and having the triangle as a self-conjugate triangle.

247. Show that the polar of a triangle for a rectangular hyperbola which circumscribes it, touches the conic which touches the three sides at the vertices of the pedal triangle; and the pole of the triangle lies on the radical axis of the circumcircle and nine-point circle of the triangle.

248. A conic passes through the vertices and centroid of a fixed triangle. Show that the pole of the triangle for the conic lies on the line at infinity, and the polar touches the maximum inscribed ellipse.

249. A conic touches the sides of a triangle and passes through its centroid. Show that the polar of the triangle for this conic is a tangent to the minimum ellipse circumscribing the triangle.

250. The foci of a conic inscribed in a triangle self-conjugate for a rectangular hyperbola are conjugate points for the r. h.

251. A parabola touches the sides of a triangle ABC, and S is its focus. The axis meets the circumcircle again in O. With O as centre the rectangular hyperbola is described for which the triangle is self-conjugate. Show that the axis of the parabola is an asymptote of the r. h.

252. Two parabolas touch the sides of a triangle, and have their foci at the extremities of a diameter of its circumcircle. Show that their axes are the asymptotes of a rectangular hyperbola for which the triangle is self-conjugate.

253. Triangles can be inscribed in a parabola (whose latus rectum is l) so as to be self-conjugate for a coaxial parabola (whose latus rectum is l'). Prove that $l' = 2l$.

254. The locus of the centre of a circle, of constant radius, circumscribed to a triangle self-conjugate for a fixed conic is a circle concentric with the conic.

255. Given three tangents and the sum of the squares of the axes, the locus of the centre of a conic is a circle.

256. A circle of given radius is inscribed in a triangle self-conjugate for a fixed conic. Prove that the locus of its centre is a concentric homothetic conic.

257. A circle α touches the sides of a triangle self-conjugate for a conic β. Show that a rectangular hyperbola having double contact with β along a tangent to α passes through the centre of the circle.

258. A circle touches a fixed straight line, and triangles can be circumscribed to it which are self-conjugate for a fixed conic. Prove that the locus of its centre is a rectangular hyperbola.

259. The orthocentre of a triangle of tangents to a rectangular hyperbola, and the centre of the circle through the points of contact are conjugate points for the r. h.

260. If the centroid of a triangle inscribed in a conic lies on a concentric homothetic conic, prove that the nine-point circle cuts orthogonally a fixed circle.

261. If two circles touch respectively the sides of two triangles self-conjugate for a conic, then their centres of similitude are conjugate points for the conic.

262. If a rectangular hyperbola has double contact with a conic a, its centre and the pole of the chord of contact are inverse points for the director circle of a.

263. A circle circumscribes triangles self-conjugate for a given conic and passes through a fixed point. Prove that its centre lies on the directrix of the parabola which has double contact with the conic at the points of contact of tangents from the fixed point.

264. Triangles are circumscribed to a central conic so as to have the same orthocentre. Prove that they have the same polar circle.

265. Two triangles are inscribed in a conic (which is not

a rectangular hyperbola) so as to have the same orthocentre. Prove that they have the same polar circle.

266. Two triangles are inscribed in a conic (which is not a circle) so that their circumcircles are concentric. Prove that they are self-conjugate for a parabola.

267. Two triangles are circumscribed to a conic, so that their circumcircles are concentric. Prove that they either have the same circumcircle or are self-conjugate for a parabola.

268. A conic which is inscribed in a triangle self-conjugate for a rectangular hyperbola and has a focus at the centre of the r. h., is a parabola.

269. A conic, with a focus at the centre of a rectangular hyperbola, circumscribes triangles self-conjugate for the r. h. Prove that the corresponding directrix touches the r. h.

270. Triangles can be inscribed in each of two conics α and β, self-conjugate for the other. Prove that the reciprocal of α for β and of β for α is the same conic γ; and α, β, γ are so related that each is the envelope of lines divided harmonically by the other two, and also the locus of points from which tangents to the other two form a harmonic pencil. Also any two of these conics are reciprocals for the third.

271. Two hyperbolas pass each through the centre of the other and determine a harmonic range on the line at infinity. Prove that the reciprocal of either, for the other, is the parabola inscribed in the quadrilateral formed by parallels, through each centre, to the asymptotes of the hyperbola passing through it.

272. A conic is inscribed in a given triangle and passes through its circumcentre. Show that its director circle touches the circumcircle and the nine-point circle of the triangle.

273. Find the locus of the centre of the conic in the last example.

274. The locus of the centre of a conic touching three given straight lines and passing through a given point is the conic touching the triangle formed by the middle points of the sides of the fixed triangle and such that, if D be the fixed point, G the centroid of the triangle and O the centre of the locus, then ODG are collinear, and $DO = \frac{3}{4} DG$.

275. If the fixed point be the centroid of the triangle, the locus is the maximum ellipse inscribed in the triangle formed by joining the middle points of the sides.

276. A circle inscribed in a triangle self-conjugate for a hyperbola cuts the hyperbola orthogonally at a point P. Show that the normal at P is parallel to an asymptote.

277. A circle is inscribed in a triangle self-conjugate for a conic and has its centre on its director circle. Prove that it touches the reciprocal of the director circle for the conic.

278. A circle, a, with centre O, is inscribed in a triangle self-conjugate for a conic, β. If P and Q be the points of contact of tangents to β from O, then the tangents from P and Q to the conic which is the reciprocal for β of its director, are also tangents to the circle a.

279. The six tangents to a conic from the vertices of a triangle cut again in twelve points which lie by sixes on four conics.

280. The six points in which a conic cuts the sides of a triangle can be joined in pairs by twelve other lines which are tangents by sixes to four conics.

281. If tangents are drawn to a parabola from two points A and B, the asymptotes of the conic through AB and the points of contact of the tangents from A and B, are parallel to the tangents to the parabola from the middle point of AB.

282. If tangents are drawn to a parabola from A and B, the conic through AB and the points of contact will be a circle, rectangular hyperbola or parabola as AB is bisected by the focus, directrix, or parabola respectively.

283. Tangents are drawn to a circle from two points on a diameter. Show that the foci of the conic touching the tangents and their chords of contact lie on the circle.

284. If tangents are drawn to a central conic from P and Q, and C be the centre and S a focus, then the conic through P, Q, and the points of contact of tangents from P, Q will be a circle if the angle PCQ is bisected internally by CS, and if $CP \cdot CQ = CS^2$.

285. The conic in the previous example will be a rectangular hyperbola if P and Q are conjugate for the director circle.

286. A point, and the orthocentre of the triangle formed by tangents from it to a conic and their chord of contact, are conjugate points for the director circle of the conic.

287. If a conic, a, pass through two points A, B and the points of contact of tangents from them to a given conic, and if β be the similarly constructed conic for two points A', B'; then if AB are conjugate for β, $A'B'$ are conjugate for a.

288. The reciprocal of the director circle of a conic, a, for a is confocal with a.

289. Along the normal to a conic at a point O are taken pairs of points PQ such that $OP \, . \, OQ$ is equal to the square of the semi-diameter parallel to the tangent at O. Show that tangents to the conic from P and Q intersect on the circle of which a diameter is the intercept made on the tangent at O by the director circle.

290. The orthocentre of a triangle formed by two tangents to a conic and their chord of contact lies on the conic. Prove that the locus of the vertex of the triangle is the reciprocal of the conic for its director circle or the reciprocal for the conic of its evolute.

291. The centre of the circle inscribed in a triangle formed by two tangents to an ellipse and their chord of contact lies on the conic. Prove that the locus of the vertex of the triangle is a hyperbola, confocal with the ellipse, and having the equi-conjugate diameters of the ellipse for its asymptotes.

292. The centre of gravity of a triangle, formed by two tangents to a conic and their chord of contact, lies on the conic. Prove that the locus of the vertex of the triangle is a concentric homothetic conic.

293. From two points BC, tangents are drawn to a fixed conic, and the sides of the two triangles formed by these two pairs of tangents and their chords of contact touch the conic a. Similarly the pairs of points CA, AB determine the conics β and γ respectively. Prove that if A lies on a, then B lies on β, and C on γ.

294. $A'B'C'$ are the middle points of the sides of a triangle ABC. Prove that the conic which is concentric with the nine-point circle of $A'B'C'$ and inscribed in $A'B'C'$ has double contact with the polar circle of ABC at the

points where the circumcircle of ABC meets the polar circle, and also has double contact with the nine-point circle of $A'B'C'$.

295. A triangle is self-conjugate for a conic. Prove that the sides of the pedal triangle touch a confocal.

296. A triangle is self-polar for a conic; show that an infinite number of triangles can be at once inscribed in the conic and circumscribed to the triangle, and vice versa.

297. If two conics α and β are related so that the poles for α of two opposite common chords lie on β, then the polars for β of two opposite common apexes touch α.

298. Of all conics inscribed in a given triangle, that for which the sum of the squares of the axes is least has its centre at the orthocentre of the triangle.

299. E, F are a pair of inverse points with respect to a circle whose centre is A; B is the harmonic conjugate of A with respect to E, F; AP, BP and the tangent at P, any point on the circle, meet the polar of E in L, M, T respectively; show that LT, TM subtend equal angles at A.

300. The connector of a pair of conjugate points with respect to a given conic passes through a fixed point, and one of the pair lies on a given straight line; show that the locus of the other is a conic, and determine six points upon the locus.

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ABBREVIATIONS

a. of h. and c. of h.	= axis and centre of homology.
r. h.	= rectangular hyperbola.
director	= director circle of central conic and directrix of parabola.
$(AB; CD)$	= intersection of the lines AB and CD.
(AB, CD)	= cross ratio of A, B, C, D.
(A, b)	= length of perpendicular from the point A to the line b.

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